K-STABILITY OF DEL PEZZO SURFACES WITH A SINGLE QUOTIENT SINGULARITY

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ABSTRACT. In this paper, we study the K-stability of del Pezzo surfaces with a single quotient singularity whose minimal resolution admits exactly two exceptional curves E_1 and E_2 with $E_1^2 = -n$, $E_2^2 = -m$ for $n, m \geq 2$.

1. INTRODUCTION

The Yau–Tian–Donaldson conjecture predicted that a smooth Fano variety X admits a Kähler–Einstein metric if and only if X is K-polystable. This conjecture was confirmed by [9, 10, 11, 38, 37]. Consequently, every K-stable Fano variety admits a Kähler–Einstein metric. A smooth del Pezzo surface S is K-polystable if and only if either $S \cong \mathbb{P}^2$ or S is isomorphic to a surface obtained by blowing-up $\mathbb{P}^2$ at k general points, where $k \geq 3$ [36, 39]. However, for a given (possibly singular) Fano variety X, determining whether X is K-(poly)stable is a delicate problem. The δ-invariant is introduced and shown to provide the criteria for K-(poly)stability. More precisely, by [3], it is shown that proving the K-stability of X amounts to proving that the δ-invariant $\delta(X)$ is greater than 1. The recent development of the Abban–Zhuang theory by [1] allows us to systematically estimate the δ-invariant. See Section 2 for more details.

The δ-invariants of smooth del Pezzo surfaces were computed in [2, 7, 32]. Although the K-stability of some Du Val del Pezzo surfaces was determined in [6, 31], the δ-invariants of all Du Val del Pezzo surfaces were recently computed in a series of papers [16, 17, 18, 19] by using the Abban–Zhuang theory. Furthermore, the K-stability of surfaces obtained by blowing-up $\mathbb{P}(1,1,n)$ at $k \leq n+4$ smooth general points is studied in [23]. More precisely, let S_n^k be a surface obtained by blowing-up k smooth general points in $\mathbb{P}(1,1,n)$. For any $n+1$ smooth general points $p_1, \ldots, p_{n+1}$ in $\mathbb{P}(1,1,n)$, there exists the unique curve C that passes through all of these points. Note that the strict transform $\tilde{C}$ of C is a (-1)-curve. By contracting the curve $\tilde{C}$, we obtain the birational morphism $\pi\colon S_n^{n+1} \to \overline{S}_n^{n+1}$. We refer the reader to [5] for details. Moreover, these surfaces are connected by blowing-up smooth general points as follows.

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$$\begin{array}{ccccccccc} S_n^0 := \mathbb{P}(1,1,n) & \longleftarrow & S_n^1 & \longleftarrow \cdots \longleftarrow & S_n^{n+1} & \longleftarrow \cdots \longleftarrow & S_n^{n+4} \\ & & & & \big\downarrow & & \\ & & & & \overline{S}_n^{n+1} & & \end{array}$$

FIGURE 1. del Pezzo surfaces with a single singularity of type $\frac{1}{n}(1,1)$

Note that $\overline{S}_n^{n+1}$ can be embedded as a degree $n+1$ hypersurface in $\mathbb{P}(1,1,1,n)$, and can also be obtained by blowing-up $\mathbb{P}^2$ at $n+1$ points on a line L and then contracting the strict transform of L. See Example 2.6 and Proposition 2.8.

Theorem 1.1 (cf. [23, Theorem B]). *Let $n \geq 4$ be a positive integer, and S a del Pezzo surface which is isomorphic to one of the surfaces in Figure 1. Then S is K-stable if and only if $S \cong S_n^{n+4}$.*

In this paper, we generalize Theorem 1.1 by allowing the blown-up points not necessarily to be in general positions.

Definition 1.2. Let $n \geq 2$ and $\pi_n \colon \tilde{S}_1 \to \mathbb{P}(1,1,n)$ a blow-up at $m \geq 2$ distinct smooth points on a line $\ell_1 \in |\mathcal{O}_{\mathbb{P}(1,1,n)}(1)|$, and L_1 the strict transform of ℓ_1. By contracting L_1, we obtain the birational morphism $\varphi_n \colon \tilde{S}_1 \to S_{n,m}^0$.

$$\mathbb{P}(1,1,n) \xleftarrow{\ \pi_n\ } \tilde{S}_1 \xrightarrow{\ \varphi_n\ } S_{n,m}^0$$

FIGURE 2. Construction of $S_{n,m}^0$

Let $S_{n,m}^k$ be the surface obtained by blowing-up at k distinct smooth general points on $S_{n,m}^0$.

We note that the surface $S_{n,m}^n$ can also be obtained by blowing-up points on the surface $\mathbb{P}(1,1,m)$. Let $\ell_2 \in |\mathcal{O}_{\mathbb{P}(1,1,m)}(1)|$ be a curve, and $\pi_m \colon \tilde{S}_2 \to \mathbb{P}(1,1,m)$ a blow-up at n distinct smooth points on ℓ_2. By contracting the strict transform L_2 of ℓ_2, we obtain the birational morphism $\varphi_m \colon \tilde{S}_2 \to S_{m,n}^0$. Let $f \colon S_{m,n}^m \to S_{m,n}^0$ be a blow-up at m distinct smooth general points on $S_{m,n}^0$. By carefully choosing the position of the points, the surface $S_{m,n}^m$ can be identified with $S_{n,m}^n$. For $n, m \geq 3$, the singular del Pezzo surfaces $S_{n,m}^{k_1}$ and $S_{m,n}^{k_2}$ can be connected by the following blow-up diagram as explained in [25, Corollary 3.5.2].

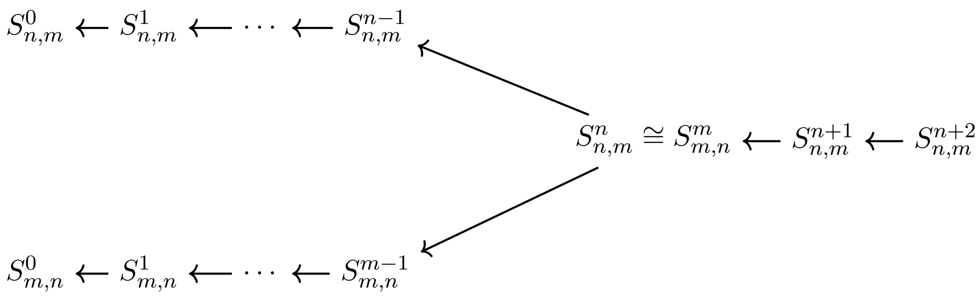

FIGURE 3. del Pezzo surfaces with a single singularity of type $\frac{1}{mn-1}(1,n)$

Remark 1.3. Assume that $n \geq m \geq 2$. The surface $S^{n+3}_{n,m}$ is still a singular del Pezzo surface with a singularity of type $\frac{1}{mn-1}(1,n)$ if and only if (n,m) is one of the following: $(2,2), (3,2)$ or $(4,2)$.

The surfaces in Figure 1 have only one exceptional curve in their minimal resolution. However, the surfaces in Figure 3 have exactly two exceptional curves in the minimal resolution. See Remark 2.4 for details.

Now, we present the main result of this paper.

Theorem 1.4. *Let S be a del Pezzo surface which is isomorphic to one of the surfaces in Figure 3 and Remark 1.3. Then the following statements hold:*

(1) S is K-stable if and only if it is isomorphic to one of the following surfaces: $S^4_{2,2}, S^5_{2,2}, S^5_{3,2}, S^6_{3,2}, S^6_{4,2}, S^7_{4,2}, S^5_{3,3}$ or $S^6_{4,3}$.

(2) S is strictly K-semistable if and only if it is isomorphic to either $S^3_{2,2}$ or $S^7_{5,2}$.

Theorem 1.4 is summarized in Table 1.

TABLE 1. K-stability of $S^k_{n,m}$

(n,m)	k	K-stability	Reference
$n+m \geq 8$	$k \leq n+2$	K-unstable	Theorem 3.2
$(2,2)$	$k \leq 2$	K-unstable	[16]
$(2,2)$	$k = 3$	strictly K-semistable	[31, 17]
$(2,2)$	$k = 4,5$	K-stable	[6, 18, 19]
$(3,2)$	$k \leq 4$	K-unstable	Theorem 3.2
$(3,2)$	$k = 5,6$	K-stable	Theorems 3.7 and 3.14
$(4,2)$	$k \leq 5$	K-unstable	Theorem 3.2
$(4,2)$	$k = 6,7$	K-stable	Theorems 3.10 and 3.17

$(3,3)$	$k \le 4$	K-unstable	Theorem 3.2
$(3,3)$	$k = 5$	K-stable	Theorem 3.24
$(4,3)$	$k \le 5$	K-unstable	Theorem 3.2
$(4,3)$	$k = 6$	K-stable	Theorem 3.27
$(5,2)$	$k \le 6$	K-unstable	Theorem 3.2
$(5,2)$	$k = 7$	strictly K-semistable	Theorem 3.21

We note that $S^{n+1}_{n,m}$ can also be obtained by running a $-K_S$-minimal model program for some smooth rational surface S with big anticanonical divisor $-K_S$. This surface S was proven to be a Mori dream space by [34]. See Proposition 2.9 and Remark 2.28 for details.

In [40], for a pair (X, Δ) such that $-(K_X + \Delta)$ is big, if there exists the anticanonical model (Z, Δ_Z) of (X, Δ), then the K-stability of (X, Δ) and (Z, Δ_Z) is the same. Hence, the K-stability of the minimal resolution of each surface $S^k_{n,m}$ is automatically obtained.

The rest of the paper is organized as follows. In Section 2, we first recall the basic notation for the quotient singularities of surfaces. In addition, we recall the definitions of notions related to K-stability, including the α, β and δ-invariants, and their criteria for K-stability. Moreover, we also recall the notion of potential pairs. Section 3 is devoted to proving the main result. The strategy first involves proving that $S^k_{n,m}$ is K-unstable for all $k \le n+2$ whenever $m+n \ge 8$, using Theorem 2.17. When $m + n \le 7$, using the Abban–Zhuang theory, we show that the surfaces $S^4_{2,2}, S^5_{2,2}, S^5_{3,2}, S^6_{3,2}$, $S^6_{4,2}, S^7_{4,2}, S^5_{3,3}$ and $S^6_{4,3}$ are K-stable. For $S^7_{5,2}$, by additionally showing that the automorphism group $\mathrm{Aut}(S^7_{5,2})$ is finite, we prove that $S^7_{5,2}$ is strictly K-semistable.

2. Preliminaries

Throughout this paper, we work over the complex number field $\mathbb{C}$. We first recall the basic definition of the cyclic quotient singularity and of the Hirzebruch–Jung fraction.

2.1. Cyclic quotient singularities of surfaces.

Definition 2.1. Let r be a positive integer, and $\varepsilon \coloneqq \exp\left(\frac{2\pi i}{r}\right)$ the primitive r-th root of unity. Consider an action of the cyclic group $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{C}^2$ defined by $(x,y) \mapsto (\varepsilon x, \varepsilon^a y)$, where a is an integer. We denote the *singularity type* of the cyclic quotient singularity $p \in \mathbb{C}^2/(\mathbb{Z}/r\mathbb{Z})$ by $\frac{1}{r}(1,a)$.

Let S be an algebraic surface. We say that a singular point $q \in S$ is *of type* $\frac{1}{r}(1,a)$ if the point q is locally analytically isomorphic to $p \in \mathbb{C}^2/(\mathbb{Z}/r\mathbb{Z})$.

Definition 2.2. Let r and a be coprime integers with $r > a > 0$. The *Hirzebruch–Jung fraction* of $\frac{r}{a}$ is the expression

$$\frac{r}{a} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \dots}} = [a_1, \dots, a_k].$$

Proposition 2.3. [24, p. 89] *Let S be an algebraic surface and $p \in S$ a cyclic quotient singularity of type $\frac{1}{r}(1, a)$. Let $\frac{r}{a} = [a_1, \dots, a_k]$ be the Hirzebruch–Jung fraction. Then the minimal resolution $\pi\colon \tilde{S} \to S$ has exceptional curves $E_1, \dots, E_k$ such that $E_i^2 = -a_i$.*

Remark 2.4. The surfaces S_n^k in Figure 1 have a single quotient singular point of type $\frac{1}{n}(1, 1)$. Since the Hirzebruch–Jung fraction of $\frac{n}{1}$ is $[n]$, the minimal resolution has the unique exceptional curve E with $E^2 = -n$ by Proposition 2.3.

The surfaces $S_{n,m}^k$ in Figure 3 have the unique quotient singular point of type $\frac{1}{mn-1}(1, n)$. In this case, the Hirzebruch–Jung fraction of $\frac{mn-1}{n}$ is $[m, n]$. Hence, by Proposition 2.3, the minimal resolution has exactly two exceptional curves E_1 and E_2 with $E_1^2 = -n$ and $E_2^2 = -m$.

2.2. **Embedding models.** In this subsection, we note that some of the surfaces in Figures 1 and 3 can be embedded in some weighted projective space as hypersurfaces or complete intersections. Moreover, we will show that the surfaces $\tilde{S}_n^{n+1}$ and $S_{n,m}^{n+1}$ can be obtained from the projective plane $\mathbb{P}^2$ as in Propositions 2.8 and 2.9.

By [8, Subsection 4.4], we have the following proposition that will be used in the proof of the main theorem.

Proposition 2.5. [8, Subsection 4.4] *Let n be a positive integer. The surface S_{2n-1}^{2n+3} can be embedded in $\mathbb{P}(1, 1, n, n, 2n-1)$ as a complete intersection of two degree $2n$ hypersurfaces. The surface S_{2n}^{2n+4} can be embedded in $\mathbb{P}(1, 1, n, n+1)$ as a degree $2n+2$ hypersurface.*

By [33], one can compute the Hilbert series.

Example 2.6. The Hilbert series of $\overline{S}_n^{n+1}$ is

$$\frac{1 - t^{n+1}}{(1-t)^3(1-t^n)},$$

which implies that $\overline{S}_n^{n+1}$ can be embedded as a degree $n+1$ hypersurface in $\mathbb{P}(1, 1, 1, n)$.

Example 2.7. The Hilbert series of $S_{n,m}^0$ is

$$\frac{1 + t^n + t^{2n} + \dots + t^{n(m-1)}}{(1-t)^2(1-t)^{nm-1}} = \frac{1 - t^{nm}}{(1-t)^2(1-t^n)(1-t^{nm-1})},$$

which implies that $S_{n,m}^0$ can be embedded as a degree nm hypersurface in $\mathbb{P}(1, 1, n, nm-1)$.

However, in general, we do not know whether $S_{n,m}^k$ can be embedded as a hypersurface or a complete intersection in the weighted projective space.

Smooth rational surfaces with a big anticanonical divisor are Mori dream spaces [34]. In particular, they proved that blowing-up $\mathbb{P}^2$ at k points in a reducible cubic is a Mori dream space under some condition on the number of points on each irreducible component [34, Theorem

4.3]. One can prove that the surfaces $\overline{S}_n^{n+1}$ and $S_{n,m}^{n+1}$ are anticanonical minimal models of such smooth rational surfaces with a big anticanonical divisor.

Proposition 2.8. *Let L be a line on $\mathbb{P}^2$, and $p_1, \ldots, p_{n+1}$ the $n+1$ distinct points on L. Let $\varphi\colon \tilde{S} \to \mathbb{P}^2$ be the blow-up of these points. By contracting the strict transform $\tilde{L}$ of L, we obtain the birational morphism $\pi\colon \tilde{S} \to S$. Then S is isomorphic to the surface $\overline{S}_n^{n+1}$ in Figure 1.*

Proof. Let $\pi_1\colon \tilde{S}_n^{n+1} \to \overline{S}_n^{n+1}$ be the minimal resolution. Then one can see that $\tilde{S}_n^{n+1}$ can be obtained by first blowing-up the n-th Hirzebruch surface $\mathbb{F}_n$ at $n+1$ distinct points $q_1, \ldots, q_{n+1}$ on the section C such that $C^2 = n$, and then contracting the strict transform $\tilde{C}$ of C. Let F_i be the fibres passing through each point q_i, and $\tilde{F}_i$ the strict transform of F_i. Then we have $\tilde{F}_i^2 = -1$ and the Picard number $\rho(\tilde{S}_n^{n+1})$ is equal to $2 + (n+1) - 1 = n+2$. By contracting the (-1)-curves $\tilde{F}_1, \ldots, \tilde{F}_{n+1}$, we obtain a smooth projective surface S' with $\rho(S') = 1$. By the classification of surfaces, we conclude that $S' \cong \mathbb{P}^2$. Hence, S is isomorphic to $\overline{S}_n^{n+1}$. □

Proposition 2.9. *Let L_1 and L_2 be two distinct lines on $\mathbb{P}^2$, $\{p_1, \ldots, p_{n+1}\}$ and $\{q_1, \ldots, q_{m+1}\}$ the sets of distinct points on L_1 and L_2, respectively. Suppose that none of these points is the intersection point of L_1 and L_2. Let $\pi\colon S_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $n+m+2$ points $p_1, \ldots, p_{n+1}, q_1, \ldots, q_{m+1}$ and $\varphi\colon S_1 \to S_2$ the birational morphism obtained by contracting the $(-n)$ and $(-m)$-curves. Then S_2 is isomorphic to $S_{n,m}^{n+1}$.*

Proof. Let $\pi_1\colon \widetilde{S}_{n,m}^{n+1} \to S_{n,m}^{n+1}$ be the minimal resolution. Then one can see that $\widetilde{S}_{n,m}^{n+1}$ can be obtained by blowing-up the n-th Hirzebruch surface $\mathbb{F}_n$ at m points $r_1, \ldots, r_m$ on a fibre F and $n+1$ points $s_1, \ldots, s_{n+1}$ on the section C such that $C^2 = n$. Let E_i be the exceptional curve over the point r_i, and F_j the fibre passing through each point s_j. Let $\tilde{C}$ and $\tilde{F}_j$ be the strict transforms of C and F_j, respectively. Then we have $\tilde{C}^2 = \tilde{F}_j^2 = -1$. By contracting all the (-1)-curves E_i, F_j and $\tilde{C}$, we obtain the birational morphism $\pi_2\colon \widetilde{S}_{n,m}^{n+1} \to Z$, which is the minimal model of $\widetilde{S}_{n,m}^{n+1}$. Since Z is a smooth projective surface and the Picard number $\rho(Z)$ is equal to $n+m+3-(n+1)-m-1 = 1$, by the classification of surfaces, we can conclude that $Z \cong \mathbb{P}^2$. Moreover, if we let E_i' and E_j'' be the exceptional curves over each point p_i and q_j, respectively, then up to permutation, we can identify $E_i' \cong E_t$ and $E_j'' \cong F_u$ or $E_j'' \cong \tilde{C}$. Therefore, one can see that $X \cong \widetilde{S}_{n,m}^{n+1}$ and hence, the surface Y is isomorphic to $S_{n,m}^{n+1}$. □

2.3. **K-stability.** In this subsection, we recall the definitions and various criteria related to K-stability. We start this subsection by presenting the definition of K-stability.

Let X be a $\mathbb{Q}$-Fano variety, i.e., a projective variety such that $-K_X$ is an ample $\mathbb{Q}$-divisor with at worst klt singularities.

Definition 2.10. A *test configuration* $(\mathcal{X}, \mathcal{L})$ of the pair $(X, -K_X)$ consists of

(1) a normal variety $\mathcal{X}$ with a $\mathbb{G}_m$-action,
(2) a flat $\mathbb{G}_m$-equivariant morphism $\pi\colon \mathcal{X} \to \mathbb{P}^1$, where $\mathbb{G}_m$ acts on $\mathbb{P}^1$ by

$$(t, [x:y]) \mapsto [tx:y], \text{ and}$$

(3) a $\mathbb{G}_m$-invariant π-ample $\mathbb{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$ and a $\mathbb{G}_m$-equivariant isomorphism

$$\left(\mathcal{X} \setminus \pi^{-1}([0:1]), \mathcal{L}|_{\mathcal{X}\setminus\pi^{-1}([0:1])}\right) \cong \left(X \times \left(\mathbb{P}^1 \setminus \{[0:1]\}\right), \pi_1^* L\right),$$

where π_1 is the projection to the first factor.

A test configuration $(\mathcal{X}, \mathcal{L})$ is said to be *trivial* if there is a $\mathbb{G}_m$-equivariant isomorphism

$$\left(\mathcal{X} \setminus \mathcal{X}_\infty, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_\infty}\right) \cong \left(X \times \left(\mathbb{P}^1 \setminus \infty, \pi_1^* L\right)\right).$$

We say a test configuration $(\mathcal{X}, \mathcal{L})$ is *of product type* if there is an isomorphism

$$\mathcal{X} \setminus \mathcal{X}_\infty \cong X \times (\mathbb{P}^1 \setminus \infty).$$

The *Donaldson–Futaki invariant* $\mathrm{DF}(\mathcal{X}; \mathcal{L})$ of the test configuration $(\mathcal{X}, \mathcal{L})$ is defined as

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) \coloneqq \frac{1}{(-K_X)^n}\left(\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} + \frac{n}{n+1}\mathcal{L}^{n+1}\right),$$

where $n = \dim X$.

Definition 2.11. A Fano variety X is said to be *K-semistable* if $\mathrm{DF}(\mathcal{X}; \mathcal{L}) \geq 0$ for every test configuration $(\mathcal{X}, \mathcal{L})$, and it is called *K-stable* if $\mathrm{DF}(\mathcal{X}; \mathcal{L}) > 0$ for every nontrivial test configuration $(\mathcal{X}, \mathcal{L})$. We say a Fano variety X is *K-polystable* if it is K-semistable and

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) = 0 \Longleftrightarrow (\mathcal{X}, \mathcal{L}) \text{ is of product type.}$$

Next, we recall the definitions of the α, β and δ-invariants, which give criteria for K-stability. The *α-invariant* $\alpha(X)$ is defined in [35] as

$$\alpha(X) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

Theorem 2.12. [35] *Let X be a $\mathbb{Q}$-Fano variety of dimension n. If $\alpha(X) > \frac{n}{n+1}$, then X admits a Kähler–Einstein metric.*

For a prime divisor E over X, there exists a projective birational morphism $\varphi\colon Y \to X$ such that E is a prime divisor on Y. The *center* $C_X(E)$ of E is the image $\varphi(E)$ of E on X. The *log discrepancy* of X along E is defined as

$$A_X(E) \coloneqq 1 + \mathrm{ord}_E(K_Y - \varphi^*(K_X)).$$

More generally, we define the log discrepancy for a log pair (X, Δ), i.e., Δ is an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. For a projective birational morphism $\varphi\colon Y \to X$, write $K_Y + \Delta_Y = \varphi^*(K_X + \Delta)$ for some divisor Δ_Y on Y, and let E be a prime divisor on Y. Define the log discrepancy $A_{X,\Delta}(E)$ of E with respect to (X, Δ) as

$$A_{X,\Delta}(E) \coloneqq 1 + \mathrm{ord}_E(K_Y - \varphi^*(K_X + \Delta)).$$

For a pseudoeffective divisor E over X, the *pseudoeffective threshold* $\tau_X(E)$ of E with respect to $-K_X$ is defined as

$$\tau_X(E) \coloneqq \sup\{\tau \in \mathbb{R}_{\geq 0} \mid \mathrm{vol}(\varphi^*(-K_X) - tE) > 0\}.$$

For a prime divisor E over X, the *S-invariant* $S_X(E)$ is defined as

$$S_X(E) \coloneqq \frac{1}{\mathrm{vol}(-K_X)} \int_0^{\tau_X(E)} \mathrm{vol}(\varphi^*(-K_X) - tE)\mathrm{d}t,$$

and the *β-invariant* $\beta_X(E)$ is defined as

$$\beta_X(E) \coloneqq A_X(E) - S_X(E).$$

We will often omit X and simply write $\tau(-)$, $S(-)$ and $\beta(-)$ if no confusion is likely to arise.

We have the following valuative criteria for K-stability.

Theorem 2.13. [4, 21, 28] *Let X be a $\mathbb{Q}$-Fano variety. Then X is*

(1) K-semistable if and only if $\beta_X(E) \geq 0$ for all prime divisors E over X.

(2) K-stable if and only if $\beta_X(E) > 0$ for all prime divisors E over X.

Let m be a positive integer, and $\{s_1, \dots, s_{N_m}\}$ any basis of $H^0(X, -mK_X)$. Let $D_1, \dots, D_{N_m}$ be the corresponding divisors. Then the divisor

$$D = \frac{1}{mN_m}(D_1 + \cdots + D_{N_m}) \sim_{\mathbb{Q}} -K_X$$

is called an *m-basis type* of $-K_X$.

Let

$$\delta_m(X) \coloneqq \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \text{ of } m\text{-basis type} \end{array} \right\}.$$

The *δ-invariant* $\delta(X)$ was first defined as $\delta(X) \coloneqq \limsup_{m\to\infty} \delta_m(X)$, and later it was shown that the lim sup is in fact a lim [3, Theorem 4.4]. By [3, Theorem C], we have

$$\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)},$$

where the infimum is taken over all prime divisors E over X. The *local δ-invariant* $\delta_p(X)$ of X at a point $p \in X$ is defined as

$$\delta_p(X) \coloneqq \inf_E \frac{A_X(E)}{S_X(E)},$$

where the infimum is taken over all prime divisors E over X such that $p \in C_X(E)$. We note that $\delta(X) = \inf_{p\in X} \delta_p(X)$. The definition of the δ-invariant for a Fano variety can be generalized to a pair (X, Δ) such that $-(K_X + \Delta)$ is big [40].

The following theorem explains why the δ-invariant is often called the *stability threshold*.

Theorem 2.14. [3, Theorem B] *Let X be a $\mathbb{Q}$-Fano variety. Then the following hold:*

(1) X is K-semistable if and only if $\delta(X) \geq 1$.

(2) X is uniformly K-stable if and only if $\delta(X) > 1$.

If the automorphism group is finite, then K-stability and K-polystability are equivalent conditions.

Theorem 2.15. [2, Corollary 1.5] *If $\mathrm{Aut}(X)$ is finite, then X is K-stable if and only if it is K-polystable.*

Note that we have the following implications.

$$X \text{ is K-stable} \Rightarrow X \text{ is K-polystable} \Rightarrow X \text{ is K-semistable}$$

Thus, if $\delta(X) = 1$ and $\mathrm{Aut}(X)$ is finite, then X is K-semistable but not K-polystable. In this case, we will say that X is *strictly K-semistable*.

By [3], one can also define the α-invariant as

$$\alpha(X) = \inf_E \frac{A_X(E)}{\tau_X(E)},$$

where the infimum is taken over all prime divisors E over X. Then we have the following inequalities which give a lower and upper bound of $\delta(X)$ with respect to $\alpha(X)$.

Theorem 2.16. [3, Theorem A] *Let X be a $\mathbb{Q}$-Fano variety of dimension n. Then we have the following inequalities*

$$\left(\frac{n+1}{n}\right)\alpha(X) \leq \delta(X) \leq (n+1)\alpha(X).$$

The following theorem plays a crucial role in proving Theorem 3.2.

Theorem 2.17. [29, Theorem 3] *Let G be a finite group, X a $\mathbb{Q}$-Fano variety of dimension n and $p \in X$ a quotient singularity with local analytic model $\mathbb{C}^n/G$. If X is K-semistable, then we have*

$$(-K_X)^n \leq \frac{(n+1)^n}{|G|}.$$

We will use the following lemma to prove that $\delta_p(S)$ is greater than 1 for a smooth point p.

Lemma 2.18. [26, Proposition 9.5.13] *Let S be a projective surface with at worst Du Val singularities and D an effective $\mathbb{Q}$-divisor on S. If (S, D) is not log canonical at a smooth point $p \in S$, then* $\mathrm{mult}_p(D) > 1$.

2.4. **Abban–Zhuang theory for surfaces.** Due to the development of the Abban–Zhuang theory [1], determining the K-stability has become more feasible for various classes of varieties. For a variety X and a point $p \in X$, the Abban–Zhuang theory gives a lower bound for the local δ-invariant $\delta_p(X)$. In this subsection, we recall how to estimate the δ-invariant for surfaces by using Abban–Zhuang theory.

Let S be a del Pezzo surface with at worst klt singularities and E a prime divisor over S, i.e., there exists a projective birational morphism $\varphi\colon \tilde{S} \to S$ such that E is a prime divisor on $\tilde{S}$. The $S(W^E_{\bullet,\bullet}; q)$ invariant for a point $q \in E$ can be calculated as follows.

Theorem 2.19 (cf. [2, Corollary 1.109])**.** *Let S be a del Pezzo surface with at worst klt singularities and $\varphi\colon \tilde{S} \to S$ a projective birational morphism such that E is a prime divisor on $\tilde{S}$. Let $\varphi^*(-K_S) - tE = P(t) + N(t)$ be the Zariski decomposition. For each point $q \in E$, we let*

$$\begin{aligned} h(t) &\coloneqq (P(t) \cdot E) \cdot \mathrm{ord}_q(N(t)|_E) + \int_0^\infty \mathrm{vol}_E(P(t)|_E - tq)\mathrm{d}t \\ &= (P(t) \cdot E)(N(t) \cdot E)_q + \frac{1}{2}(P(t) \cdot E)^2, \end{aligned}$$

where $(N(t) \cdot E)_q$ is the local intersection number of $N(t)$ and E at the point q.

Then we have

$$S(W^E_{\bullet,\bullet}; q) = \frac{2}{(-K_S)^2} \int_0^{\tau(E)} h(t)\mathrm{d}t,$$

where $\tau(E)$ is the pseudoeffective threshold of E with respect to $-K_S$.

Suppose that φ is a blow-up of S at a point p. Then we have $(K_{\tilde{S}}+E)|_E = K_E+\Phi$ for some divisor Φ on E, which we call the *different*. Then we have the following lower bound of the local δ-invariant.

Theorem 2.20 (cf. [1, Theorem 3.2])**.** *Let S be a del Pezzo surface with at worst klt singularities and $\varphi\colon \tilde{S}\to S$ a blow-up of S at a point $p\in S$. Let E be the exceptional curve over the point p. Then we have the following inequality:*

$$\delta_p(S)\geq \min\left\{\frac{A_S(E)}{S_S(E)}, \inf_{q\in E}\frac{A_{E,\Phi}(q)}{S(W^E_{\bullet,\bullet};q)}\right\}.$$

2.5. **Potential pair.** Let Y be a smooth projective variety and D a big divisor on Y. Let E be a prime divisor on Y. We define the *asymptotic divisorial valuation* $\sigma_E(D)$ of D along E as $\inf\{\mathrm{mult}_E(D')\mid 0\leq D'\sim_{\mathbb{Q}} D\}$. If D is only pseudoeffective, then we define $\sigma_E(D):=\lim_{\epsilon\to 0^+}\sigma_E(D+\epsilon A)$, where A is an ample divisor on Y. It is well known that this definition is independent of the choice of an ample divisor.

This definition naturally extends to normal projective varieties as follows. Let X be a normal projective variety and D a pseudoeffective divisor on X. Let E be a prime divisor over X. We define the asymptotic divisorial valuation $\sigma_E(D):=\sigma_E(f^*D)$, where $f\colon Y\to X$ is a projective birational morphism from a smooth projective variety Y such that E is a prime divisor on Y. This definition is independent of the choice of f as shown in [30, Theorem III.5.16].

For a normal projective variety X and a pseudoeffective divisor D, we define the *divisorial Zariski decomposition* as follows. We define the negative part $N(D):=\sum_E \sigma_E(D)D$, where the sum is taken over all prime divisors E over X, and $P(D):=D-N(D)$ is defined as the positive part. We note that $\sigma_E(D)>0$ for only finitely many prime divisors. The decomposition $D=P(D)+N(D)$ is called the *divisorial Zariski decomposition*. When $\dim X\geq 3$, the positive part is not necessarily nef, but only movable, unlike the Zariski decomposition for surfaces. If the positive part is nef, then we simply call it the Zariski decomposition. See [30] for more details.

Definition 2.21. [15, Definitions 3.1 and 3.2] Let (X,Δ) be a pair such that $-(K_X+\Delta)$ is pseudoeffective. For a prime divisor E over X, we define the *potential discrepancy* $\overline{a}(E;X,\Delta)$ of (X,Δ) along E as

$$\overline{a}(E;X,\Delta):=A_{X,\Delta}(E)-\sigma_E(D).$$

Then we say that the pair (X,Δ) is *potentially klt* or *pklt* for short (resp. *potentially lc* or *plc* for short) if $\inf_E \overline{a}(E;X,\Delta)>0$ (resp. ≥ 0), where the infimum is taken over all prime divisors E over X.

The notion of potential pairs was first defined and studied by [15]. See also [12, 13, 14, 27] for related results. One of the main properties of potential pairs is that one can bound the singularities of the resulting pairs of $-(K_X+\Delta)$-minimal model program (MMP for short).

We briefly explain what $-(K_X+\Delta)$-MMP and $-(K_X+\Delta)$-minimal model of a pair (X,Δ) is.

Definition 2.22. Let X be a normal projective variety and D a pseudoeffective divisor on X. A birational contraction $\varphi\colon X \dashrightarrow Y$ is called a *D-negative* if $\varphi_* D$ is $\mathbb{R}$-Cartier and there exists a common resolution $(p,q)\colon W \to X \times Y$ such that

$$p^* D = q^* \varphi_* D + E,$$

where E is an effective q-exceptional divisor whose support $\mathrm{Supp}(E)$ contains the support of all the strict transforms of the φ-exceptional divisors.

As a special case of D-negative contraction, we say $\varphi\colon X \dashrightarrow Y$ is a D-minimal model if φ is a D-negative contraction and $\varphi_* D$ is nef. The composition of D-negative contractions is called D-minimal model program (D-MMP for short). When $D = -(K_X + \Delta)$ is pseudoeffective, it is called a $-(K_X+\Delta)$-minimal model or $-(K_X+\Delta)$-MMP.

Definition 2.23. A rational map $f\colon X \dashrightarrow Y$ is called the *anticanonical model* of (X,Δ) if Y is a normal projective variety and there is an ample divisor A on Y such that if $(p,q)\colon W \to X \times Y$ is a resolution of the indeterminacy of f, then q is a contraction morphism with $-p^*(K_X+\Delta) \sim_{\mathbb{R}} q^*A + E$, where $E \geq 0$ is contained in the fixed part of $|-p^*(K_X+\Delta)|_{\mathbb{R}}$.

Theorem 2.24. [15, Proposition 3.11] *Let (X,Δ) be a pair such that $-(K_X+\Delta)$ is pseudoeffective. Suppose that $\varphi\colon (X,\Delta) \dashrightarrow (Y,\Delta_Y)$ is a $-(K_X+\Delta)$-minimal model. Then (Y,Δ_Y) is klt (resp. lc) if and only if (X,Δ) is pklt (resp. plc).*

However, being a pklt pair does not guarantee that we can run an $-(K_X+\Delta)$-MMP in general. On the other hand, if (X,Δ) is a pklt pair such that $-(K_X+\Delta)$ is big, then X is a variety of Fano type [15, Theorem 5.1] and hence, we can run the $-(K_X+\Delta)$-MMP.

In the surface case, since the notions of movable and nef divisors coincide, the notions of pklt and klt are the same.

Proposition 2.25. [15, Lemma 3.5, Corollary 5.2] *Let S be a surface such that $-K_S$ is big, and $-K_S = P + N$ the Zariski decomposition. If (S,N) is a klt pair, then S is of Fano type and $(S,0)$ is a pklt pair. In particular, by contracting all the curves in* $\mathrm{Supp}(N)$*, we obtain the $-K_S$-minimal model.*

For a klt pair (X,Δ) such that $-(K_X+\Delta)$ is big, if one can run a $-(K_X+\Delta)$-MMP and obtain the anticanonical model (Z,Δ_Z) of (X,Δ), then the K-stability of the pairs (X,Δ) and (Z,Δ_Z) coincides, as shown by the following theorem.

Theorem 2.26. [40, Theorem 1.2] *Let (X,Δ) be a klt pair such that $-(K_X+\Delta)$ is big. Suppose that there exists the anticanonical model (Z,Δ_Z) of (X,Δ). Then (X,Δ) is K-semistable (resp. K-stable, uniformly K-stable) if and only if (Z,Δ_Z) is K-semistable (resp. K-stable, uniformly K-stable).*

Proposition 2.27. *Let S_1 and S_2 be surfaces as in Proposition 2.9. Then the blow-down $\varphi\colon S_1 \to S_2$ is a $-K_{S_1}$-minimal model. In fact, S_2 is the anticanonical model of S_1.*

Proof. On S_1, let $\tilde{L}_n$ and $\tilde{L}_m$ be the strict transforms of L_1 and L_2, respectively. The anticanonical divisor $-K_{S_1}$ can be written as

$$-K_{S_1} = L + \tilde{L}_n + \tilde{L}_m,$$

where L is the pullback of the hyperplane class on $\mathbb{P}^2$. The positive and negative parts of the Zariski decomposition of $-K_{S_1}$ are as follows:

$$P = L + \frac{m}{mn-1}\tilde{L}_n + \frac{n}{mn-1}\tilde{L}_m \text{ and } N = \frac{mn-m-1}{mn-1}\tilde{L}_n + \frac{mn-n-1}{mn-1}\tilde{L}_m.$$

Since (S_1, N) is a klt pair, it is indeed a pklt pair. In addition, $(S_1, 0)$ is also a pklt pair by Proposition 2.25. Moreover, since S_1 is a Mori dream space, we can run a $-K_{S_1}$-minimal model program which can be obtained by contracting $\tilde{L}_n$ and $\tilde{L}_m$. In addition, since $-K_{S_2}$ is ample by Proposition 2.9, S_2 is the anticanonical model of S_1, which has at worst klt singularities. □

Remark 2.28. By Proposition 2.9, the surfaces $S^k_{n,m}$ with $k \geq n+1$ can be obtained by running an anticanonical MMP. More precisely, since they are singular del Pezzo surfaces, they are in fact the anticanonical models of surfaces obtained by blowing-up $\mathbb{P}^2$ in special configurations as in [34]. Moreover, by Theorem 2.26, the K-stability of the minimal resolutions of the surfaces $S^k_{n,m}$ is the same.

3. Main results and Proofs

In this section, we prove Theorem 1.4. Let us first show that when $n+m \geq 8$, the surfaces $S^{k_1}_{n,m}$ and $S^{k_2}_{m,n}$ in Figure 3 are K-unstable by using Theorem 2.17. Since we have the isomorphism $S^n_{n,m} \cong S^m_{m,n}$, we need to consider the cases where $k_1 \leq n+2$ and $k_2 \leq m-1$.

3.1. Case: $n+m \geq 8$.

In this subsection, we prove that the surfaces in Figure 3 and Remark 1.3 are K-unstable for all $k \leq n+3$ whenever $n+m \geq 8$. Without loss of generality, we may assume that $n \geq m$.

Lemma 3.1. *Let (n, m, k_1) and (m, n, k_2) be triples which appear in Figure 3 and Remark 1.3. The inequality*

$$n + 2 - k_1 + \frac{m+n+2}{mn-1} \leq \frac{9}{mn-1}$$

holds if and only if the triple (n, m, k_1) is contained in the following set:

(A) $\{(2,2,3),(2,2,4),(2,2,5),(3,2,5),(3,2,6),(4,2,6),(4,2,7),(5,2,7),(3,3,5),(4,3,6)\}$.

Also, the inequality

$$m + 2 - k_2 + \frac{m+n+2}{mn-1} \leq \frac{9}{mn-1}$$

does not hold.

Proof. The first inequality is equivalent to

$$n + 2 + \frac{m+n-7}{mn-1} \leq k_1.$$

For $k_1 \leq n+2$, this inequality has a solution if and only if $\frac{m+n-7}{mn-1} \leq 0$. We can easily find all possible values of k_1 for which $m+n \leq 7$ by direct calculation. For case $k_1 = n+3$, we can also check that the three possible cases in Remark 1.3 satisfy the inequality. Therefore, we obtain the set A. The second inequality is equivalent to

$$m + 2 + \frac{m + n - 7}{mn - 1} \le k_2.$$

Since $S^n_{n,m} \cong S^m_{m,n}$, we only need to consider the case $k_2 \le m - 1$. Then the inequality has solutions if and only if $\frac{m+n-7}{mn-1} \le -3$, which is impossible. Thus, the second inequality cannot hold. □

Theorem 3.2. *Let $S^{k_1}_{n,m}$ and $S^{k_2}_{m,n}$ be the surfaces as in Figure 3 and Remark 1.3. The surface $S^{k_2}_{m,n}$ is K-unstable for all $n, m \ge 2$ and $0 \le k_2 \le m - 1$, and the surface $S^{k_1}_{n,m}$ is K-unstable except when the triple (n, m, k_1) is contained in the set A. In particular, if $m + n \ge 8$, then the surface $S^{k_1}_{n,m}$ is K-unstable for all $k_1 \le n + 2$.*

Proof. Let $S_1 := S^{k_1}_{n,m}$ and $S_2 := S^{k_2}_{m,n}$ for simplicity. We note that

$$(-K_{S_1})^2 = n + 2 - k_1 + \frac{m + n + 2}{mn - 1} \text{ and } (-K_{S_2})^2 = m + 2 - k_2 + \frac{m + n + 2}{mn - 1}.$$

The singular points $p \in S_1$ and $q \in S_2$ are locally analytically isomorphic to $\mathbb{C}^2/G_1$ and $\mathbb{C}^2/G_2$ with $|G_1| = |G_2| = mn - 1$. Note that G_1 acts on $\mathbb{C}^2$ by $(x, y) \mapsto (\varepsilon x, \varepsilon^n y)$ and G_2 acts on $\mathbb{C}^2$ by $(x, y) \mapsto (\varepsilon x, \varepsilon^m y)$, where $\varepsilon := \exp\left(\frac{2\pi i}{mn-1}\right)$ is the $(mn - 1)$th primitive root of unity. If S_1 and S_2 are K-semistable, then by Theorem 2.17, we have

$$n + 2 - k_1 + \frac{m + n + 2}{mn - 1} \le \frac{9}{mn - 1} \text{ and } m + 2 - k_2 + \frac{m + n + 2}{mn - 1} \le \frac{9}{mn - 1}.$$

By Lemma 3.1, we conclude the proof. □

3.2. **Case:** $m + n \le 7$. Without loss of generality, we may assume that $n \ge m$. By Theorem 3.2, we only need to consider the surfaces $S^k_{n,m}$ for which the triple (n, m, k) is contained in the solution set A.

Remark 3.3. The δ-invariants of the surfaces $S^k_{2,2}$ are computed in [16, 17, 18, 19]. The singularity of $S^k_{2,2}$ is of type $\frac{1}{3}(1, 2)$, which is a Du Val singularity. By [16, 17, 18, 19], S is K-unstable except for $S^3_{2,2}, S^4_{2,2}, S^5_{2,2}$. Note that $S^3_{2,2}$ is strictly K-semistable and $S^4_{2,2}$ and $S^5_{2,2}$ are K-stable. The K-stability of some Du Val del Pezzo surfaces had already been established in [6, 31].

The remaining cases are the surfaces $S^5_{3,2}, S^6_{3,2}, S^6_{4,2}, S^7_{4,2}, S^7_{5,2}, S^5_{3,3}$ and $S^6_{4,3}$. The rest of this paper is devoted to proving that they are all K-stable except $S^7_{5,2}$. With some additional effort, we will show that the surface $S^7_{5,2}$ is strictly K-semistable.

3.2.1. *Case:* $S^6_{3,2}$. Let $\pi\colon S \to \mathbb{P}(1, 1, 3)$ be the blow-up of $\mathbb{P}(1, 1, 3)$ at seven smooth general points $p_1, \dots, p_7$. Note that the surface S is isomorphic to a complete intersection of two hypersurfaces of degree 4 in $\mathbb{P}(1, 1, 2, 2, 3)$ by Proposition 2.5. Let L be the strict transform of the curve $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,3)}(1)|$ passing through the point p_1. Let $\pi_1\colon S_1 \to S$ be a blow-up of S at a point $p \in L$ such that $\pi(p) \ne p_1$. Then by contracting the strict transform L_1 of L, we obtain the birational morphism $\varphi\colon S_1 \to S^6_{3,2}$. For the singular point $q \in S_1$ of type $\frac{1}{3}(1, 1)$, let $\pi_2\colon S_2 \to S_1$ be the weighted blow-up with weights $(1, 1)$. We consider the following diagram.

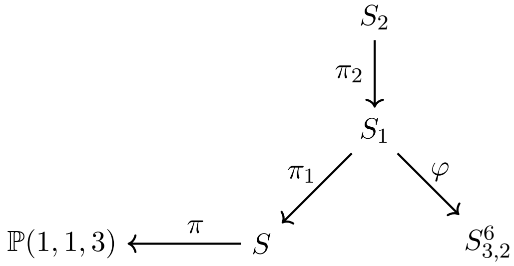

FIGURE 4

Lemma 3.4. *Let $p \in S^6_{3,2}$ be the singular point of type $\frac{1}{5}(1,3)$. Then we have $\delta_p(S^6_{3,2}) > 1$.*

Proof. Consider a curve $c \in |\mathcal{O}_{\mathbb{P}(1,1,3)}(4)|$ passing through the six points $\{p_2, \dots, p_7\}$. Let C be the strict transform of c under π. Then we have the following:

$$-K_S \equiv L + C, \quad L^2 = C^2 = -\frac{2}{3}, \quad L \cdot C = \frac{4}{3},$$
$$-K_{S_1} \equiv L_1 + C_1, \quad L_1^2 = -\frac{5}{3}, \quad C_1^2 = -\frac{2}{3} \text{ and } L_1 \cdot C_1 = \frac{4}{3},$$

where L_1 and C_1 are the strict transforms of L and C, respectively. We also have

$$-K_{S^6_{3,2}} \equiv R_S \text{ and } (-K_{S^6_{3,2}})^2 = \frac{2}{5}$$

such that $\varphi_*(C_1) = R_S$. We obtain that $\varphi^*(R_S) = C_1 + \frac{4}{5}L_1$.

Since C_1 is a negative curve, the pseudoeffective threshold of L_1 is $\frac{4}{5}$. The positive and negative parts of the Zariski decomposition of $\varphi^*(-K_{S^6_{3,2}}) - tL_1$ are

$$P(t) = \begin{cases} \varphi^*(-K_{S^6_{3,2}}) - tL_1 & \text{if } 0 \le t \le \dfrac{3}{10}, \\ \left(\dfrac{4}{5} - t\right)(2C_1 + L_1) & \text{if } \dfrac{3}{10} \le t \le \dfrac{4}{5}, \end{cases}$$

and

$$N(t) = \begin{cases} 0 & \text{if } 0 \le t \le \dfrac{3}{10}, \\ \left(-\dfrac{3}{5} + 2t\right) C_1 & \text{if } \dfrac{3}{10} \le t \le \dfrac{4}{5}. \end{cases}$$

Then we obtain that

$$P(t)^2 = \begin{cases} \dfrac{2}{5} - \dfrac{5}{3}t^2 & \text{if } 0 \le t \le \dfrac{3}{10}, \\ \left(\dfrac{4}{5} - t\right)^2 & \text{if } \dfrac{3}{10} \le t \le \dfrac{4}{5}. \end{cases}$$

Hence, the S-invariant of L_1 is $S(L_1) = \frac{5}{2}\left(\frac{21}{200} + \frac{1}{24}\right) = \frac{11}{30}$. Also, since $K_{S_1} = \varphi^*(K_{S^6_{3,2}}) - \frac{1}{5}L_1$, $A_{S^6_{3,2}}(L_1) = \frac{4}{5}$ and $\frac{A(L_1)}{S(L_1)} = \frac{24}{11} > 1$. Moreover, we have

$$P(t)\cdot L_1 = \begin{cases} \dfrac{5}{3}t & \text{if } 0 \le t \le \dfrac{3}{10}, \\ \dfrac{4}{5} - t & \text{if } \dfrac{3}{10} \le t \le \dfrac{4}{5}. \end{cases}$$

For every smooth point $p \in L_1$, we have

$$h(t) = \frac{1}{2}\left(P(t)\cdot L_1\right)^2 = \begin{cases} \dfrac{25}{18}t^2 & \text{if } 0 \le t \le \dfrac{3}{10}, \\ \dfrac{1}{2}\left(\dfrac{4}{5} - t\right)^2 & \text{if } \dfrac{3}{10} \le t \le \dfrac{4}{5}. \end{cases}$$

Therefore, we have $S(W^{L_1}_{\bullet,\bullet}; p) = 5\left(\frac{1}{80} + \frac{1}{48}\right) = \frac{1}{6}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1 and hence, $\delta_p(S^6_{3,2}) > 1$.

Now, consider the singular point in L_1 with type $\frac{1}{3}(1,1)$. We set $\psi = \pi_2 \circ \varphi$. Then we obtain that

$$\psi^*(R_S) = \pi_2^*\left(C_1 + \frac{4}{5}L_1\right) = C_2 + \frac{4}{5}L_2 + \frac{3}{5}E,$$

where E is the π_2-exceptional divisor, C_2 and L_2 are the strict transforms of C_1 and L_1, respectively. The intersection numbers are the following:

$$L_2^2 = -2, \quad C_2^2 = -1, \quad L_2\cdot C_2 = 1, \quad E^2 = -3 \text{ and } L_2\cdot E = C_2\cdot E = 1.$$

Since the intersection matrix corresponding to the curves C_2 and L_2 is negative definite, the pseudoeffective threshold of E is $\frac{3}{5}$.

The positive and negative parts of the Zariski decomposition of $\psi^*(-K_{S^6_{3,2}}) - tE$ are

$$P(t) = \begin{cases} C_2 + \left(\dfrac{4}{5} - \dfrac{1}{2}t\right)L_2 + \left(\dfrac{3}{5} - t\right)E & \text{if } 0 \le t \le \dfrac{4}{15}, \\ \left(\dfrac{3}{5} - t\right)(3C_2 + 2L_2 + E) & \text{if } \dfrac{4}{15} \le t \le \dfrac{3}{5}, \end{cases}$$

and

$$N(t) = \begin{cases} \dfrac{1}{2}tL_2 & \text{if } 0 \le t \le \dfrac{4}{15}, \\ \left(-\dfrac{4}{5} + 3t\right)C_2 + \left(-\dfrac{2}{5} + 2t\right)L_2 & \text{if } \dfrac{4}{15} \le t \le \dfrac{3}{5}. \end{cases}$$

Thus, we obtain that

$$P(t)^2 = \begin{cases} -\dfrac{1}{10}(5t+2)(5t-2) & \text{if } 0 \le t \le \dfrac{4}{15}, \\ 2\left(\dfrac{3}{5} - t\right)^2 & \text{if } \dfrac{4}{15} \le t \le \dfrac{3}{5}. \end{cases}$$

Hence, the S-invariant of E is

$$S(E) = \frac{5}{2}\left(\frac{184}{2025} + \frac{2}{81}\right) = \frac{13}{45}.$$

Also, we have that $K_{S_2} = \psi^*(K_{S^6_{3,2}}) - \frac{1}{5}L_2 - \frac{2}{5}E$, which implies that $A_{S^6_{3,2}}(E) = \frac{3}{5}$, and $\frac{A(E)}{S(E)} = \frac{27}{13} > 1$. Moreover, we have

$$P(t)\cdot E = \begin{cases} \frac{5}{2}t & \text{if } 0 \le t \le \frac{4}{15}, \\ \frac{6}{5} - 2t & \text{if } \frac{4}{15} \le t \le \frac{3}{5}, \end{cases}$$

For the point $p \in C_2 \cap E$, we have

$$h(t) = \begin{cases} \frac{25}{8}t^2 & \text{if } 0 \le t \le \frac{4}{15}, \\ \left(\frac{6}{5} - 2t\right)\left(-\frac{4}{5} + 3t\right) + \frac{1}{2}\left(\frac{6}{5} - 2t\right)^2 & \text{if } \frac{4}{15} \le t \le \frac{3}{5}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 5\left(\frac{8}{405} + \frac{5}{81}\right) = \frac{11}{27}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^6_{3,2}) > 1$.

For the point $p \in L_2 \cap E$, we have

$$h(t) = \begin{cases} \frac{5}{4}t^2 + \frac{25}{8}t^2 & \text{if } 0 \le t \le \frac{4}{15}, \\ \left(\frac{6}{5} - 2t\right)\left(-\frac{2}{5} + 2t\right) + \frac{1}{2}\left(\frac{6}{5} - 2t\right)^2 & \text{if } \frac{4}{15} \le t \le \frac{3}{5}. \end{cases}$$

Then $S(W^E_{\bullet,\bullet}; p) = 5\left(\frac{56}{2025} + \frac{26}{405}\right) = \frac{62}{135}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^6_{3,2}) > 1$.

For $p \in E \setminus (C_2 \cup L_2)$, we have

$$h(t) = \begin{cases} \frac{25}{8}t^2 & \text{if } 0 \le t \le \frac{4}{15}, \\ \frac{1}{2}\left(\frac{6}{5} - 2t\right)^2 & \text{if } \frac{4}{15} \le t \le \frac{3}{5}. \end{cases}$$

Then $S(W^E_{\bullet,\bullet}; p) = 5\left(\frac{8}{405} + \frac{2}{81}\right) = \frac{2}{9}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^6_{3,2}) > 1$. □

Now, we show that $\delta_p(S^6_{3,2}) > 1$ for smooth points p.

Lemma 3.5. *Let S be the surface in Figure 4, and $D \equiv -K_S$ an effective divisor. Then the pair $\left(S, \frac{3}{4}D\right)$ is log canonical along the smooth locus of S.*

Proof. Suppose that the pair $(S, \frac{3}{4}D)$ is not log canonical at a smooth point $q \in S$. Let $C \in |\mathcal{O}_S(1)|$ be a curve passing through q. We first consider when C is irreducible. By [22, Lemma 4.1], C is log canonical at q. Then we can assume that C is not contained in the support of D. We have

$$1 < \operatorname{mult}_q\left(\frac{3}{4}D\right) \le \frac{3}{4}C\cdot D = 1,$$

which leads to a contradiction. Thus, C is reducible. Let $C = C_1 + C_2$. Without loss of generality, we may assume that $q \in C_1$. Since the pair (S, C) is log canonical at q, we can assume that at least one component of C is not contained in the support of D. If $C_1 \not\subseteq \mathrm{Supp}(D)$, then the inequality

$$1 < \mathrm{mult}_q\left(\frac{3}{4}D\right) \leq \frac{3}{4}C_1 \cdot D = \frac{1}{2}$$

leads to a contradiction. Thus, let $D = aC_1 + \Delta$, where a is a positive number and Δ is an effective divisor such that $C_1 \not\subseteq \mathrm{Supp}(\Delta)$. The inequality

$$\frac{2}{3} \geq D \cdot C_2 \geq aC_1 \cdot C_2 = \frac{4}{3}a$$

implies that $a \leq \frac{1}{2}$. By the inversion of adjunction formula, we obtain that

$$1 < \mathrm{mult}_q\left(\frac{3}{4}\Delta|_{C_1}\right) \leq \frac{3}{4}\Delta \cdot C_1 = \frac{3}{4}(D - aC_1) \cdot C_1 = \frac{1}{2} + \frac{1}{2}a.$$

This implies that $a > 1$, which contradicts the earlier bound $a \leq \frac{1}{2}$. Hence, the pair $(S, \frac{3}{4}D)$ is log canonical at q. □

Lemma 3.6. *Let $p \in S^6_{3,2}$ be a smooth point. Then we have $\delta_p(S^6_{3,2}) > 1$.*

Proof. We set $\lambda := \frac{3}{4}$ for simplicity. Let $D_S \equiv -K_{S^6_{3,2}}$ be an effective $\mathbb{Q}$-divisor. We have

$$K_{S_1} = \varphi^*(K_{S^6_{3,2}}) - \frac{1}{5}L_1 \text{ and } \varphi^*(D_S) = D_1 + \alpha L_1,$$

where D_1 is the strict transform of D_S and α is the multiplicity of D_S at p. Suppose that the pair $(S^6_{3,2}, \lambda D_S)$ is not log canonical at the point p. Then the pair $(S_1, \lambda D_1 + \left(\lambda\alpha + \frac{1}{5}\right) L_1)$ is not log canonical at a point $p_1 \in S_1$ such that $p_1 \notin L_1$ and $\varphi(p_1) = p$. This implies that the pair $(S, \lambda D + \left(\lambda\alpha + \frac{1}{5}\right) L)$ is not log canonical at the point $q = \pi_1(p_1)$ on S, where D and L are the pushforwards of D_1 and L_1, respectively. Then the pair $(S, \lambda D)$ is not log canonical at q. Meanwhile, since $D + \left(\alpha + \frac{1}{5}\right) L \equiv -K_S$, it follows from Lemma 3.5 that the pair $(S, \lambda(D + (\alpha + \frac{1}{5})L))$ is log canonical at q. This implies that the pair $(S, \lambda D)$ is log canonical at q, which is a contradiction. Therefore, $\alpha_p(S^6_{3,2}) \geq \frac{3}{4}$, and since $\frac{3}{2}\alpha_p(S^6_{3,2}) \leq \delta_p(S^6_{3,2})$ by Theorem 2.16, we have $\delta_p(S^6_{3,2}) > 1$. □

Theorem 3.7. *The surface $S^6_{3,2}$ is K-stable.*

Proof. By Lemmas 3.4 and 3.6, we see that the surface $S^6_{3,2}$ is K-stable. □

3.2.2. *Case: $S^7_{4,2}$.* Let $\pi\colon S' \to \mathbb{P}(1,1,4)$ be the blow-up of $\mathbb{P}(1,1,4)$ at eight smooth general points $p_1, \ldots, p_8$. Note that the surface S' is isomorphic to a degree 6 hypersurface embedded in $\mathbb{P}(1,1,2,3)$ by Proposition 2.5. Let L be the strict transform of the curve $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,4)}(1)|$ passing through the point p_1. Let $\pi_1\colon S'_1 \to S'$ be a blow-up of S' at a point $p \in L$ such that $\pi(p) \neq p_1$. Then by contracting the strict transform L_1 of L, we obtain the birational morphism $\varphi\colon S'_1 \to S^7_{4,2}$. For the singular point $q \in S'_1$ of type $\frac{1}{4}(1,1)$, let $\pi_2\colon S'_2 \to S'_1$ be the weighted blow-up with weights $(1,1)$. We consider the following diagram.

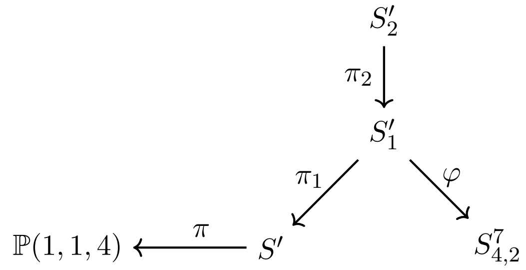

FIGURE 5

Lemma 3.8. *Let $p \in S^7_{4,2}$ be the singular point of type $\frac{1}{7}(1,4)$. Then we have $\delta_p(S^7_{4,2}) > 1$.*

Proof. Consider a curve $c \in |\mathcal{O}_{\mathbb{P}(1,1,4)}(5)|$ passing through the seven points $\{p_2, \dots, p_8\}$. Let C be the strict transform of c under π. Then we have the following:

$$-K_{S'} \equiv L + C, \quad L^2 = C^2 = -\frac{3}{4}, \quad L \cdot C = \frac{5}{4},$$
$$-K_{S'_1} \equiv L_1 + C_1, \quad L_1^2 = -\frac{7}{4}, \quad C_1^2 = -\frac{3}{4} \text{ and } L_1 \cdot C_1 = \frac{5}{4},$$

where L_1 and C_1 are the strict transforms of L and C, respectively. Then we have

$$-K_{S^7_{4,2}} \equiv R_{S'} \text{ and } (-K_{S^7_{4,2}})^2 = \frac{1}{7}$$

such that $\varphi_*(C_1) = R_{S'}$. We obtain that $\varphi^*(R_{S'}) = C_1 + \frac{5}{7}L_1$.

Since C_1 is a negative curve, the pseudoeffective threshold of L_1 is $\frac{5}{7}$. The positive and negative parts of the Zariski decomposition of $\varphi^*(-K_{S^7_{4,2}}) - tL_1$ are

$$P(t) = \begin{cases} \varphi^*(-K_{S^7_{4,2}}) - tL_1 & \text{if } 0 \le t \le \frac{4}{35}, \\ \left(\frac{5}{7} - t\right)\left(\frac{5}{3}C_1 + L_1\right) & \text{if } \frac{4}{35} \le t \le \frac{5}{7}, \end{cases}$$

and

$$N(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{4}{35}, \\ \frac{1}{3}\left(-\frac{4}{7} + 5t\right)C_1 & \text{if } \frac{4}{35} \le t \le \frac{5}{7}. \end{cases}$$

Then we obtain that

$$P(t)^2 = \begin{cases} \frac{1}{7} - \frac{7}{4}t^2 & \text{if } 0 \le t \le \frac{4}{35}, \\ \frac{1}{3}\left(\frac{5}{7} - t\right)^2 & \text{if } \frac{4}{35} \le t \le \frac{5}{7}. \end{cases}$$

Hence, the S-invariant of L_1 is $S(L_1) = 7\left(\frac{284}{18375} + \frac{3}{125}\right) = \frac{29}{105}$. Also, since $K_{S_1'} = \varphi^*(K_{S^7_{4,2}}) - \frac{2}{7}L_1$, $A(L_1) = \frac{5}{7}$ and $\frac{A(L_1)}{S(L_1)} = \frac{75}{29}$. Moreover, we have

$$P(t)\cdot L_1 = \begin{cases} \dfrac{7}{4}t & \text{if } 0 \le t \le \dfrac{4}{35}, \\ \dfrac{1}{3}\left(\dfrac{5}{7} - t\right) & \text{if } \dfrac{4}{35} \le t \le \dfrac{5}{7}. \end{cases}$$

For every smooth point $p \in L_1$, we have

$$h(t) = \frac{1}{2}\left(P(t)\cdot L_1\right)^2 = \begin{cases} \dfrac{49}{32}t^2 & \text{if } 0 \le t \le \dfrac{4}{35}, \\ \dfrac{1}{18}\left(\dfrac{5}{7} - t\right)^2 & \text{if } \dfrac{4}{35} \le t \le \dfrac{5}{7}. \end{cases}$$

Therefore, we have $S(W^{L_1}_{\bullet,\bullet}; p) = 14\left(\frac{2}{2625} + \frac{1}{250}\right) = \frac{1}{15}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1. Therefore, $\delta_p(S^7_{4,2}) > 1$.

Now, consider the singular point in L_1 with type $\frac{1}{4}(1,1)$. We set $\psi = \pi_2 \circ \varphi$. Then we obtain that

$$\psi^*(R_S) = \pi_2^*\left(C_1 + \frac{4}{5}L_1\right) = C_2 + \frac{5}{7}L_2 + \frac{3}{7}E,$$

where E is the π_2-exceptional divisor, C_2 and L_2 are the strict transforms of C_1 and L_1, respectively. The intersection numbers are the following:

$$L_2^2 = -2, \quad C_2^2 = -1, \quad L_2\cdot C_2 = 1, \quad E^2 = -4, \text{ and } L_2\cdot E = C_2\cdot E = 1.$$

Since the intersection matrix corresponding to the curves C_2 and L_2 is negative definite, the pseudoeffective threshold of E is $\frac{3}{7}$.

The positive and negative parts of the Zariski decomposition of $\psi^*(-K_{S^7_{4,2}}) - tE$ are

$$P(t) = \begin{cases} C_2 + \left(\dfrac{5}{7} - \dfrac{1}{2}t\right)L_2 + \left(\dfrac{3}{7} - t\right)E & \text{if } 0 \le t \le \dfrac{2}{21}, \\ \left(\dfrac{3}{7} - t\right)(3C_2 + 2L_2 + E) & \text{if } \dfrac{2}{21} \le t \le \dfrac{3}{7}, \end{cases}$$

and

$$N(t) = \begin{cases} \dfrac{1}{2}tL_2 & \text{if } 0 \le t \le \dfrac{2}{21}, \\ \left(-\dfrac{2}{7} + 3t\right)C_2 + \left(-\dfrac{1}{7} + 2t\right)L_2 & \text{if } \dfrac{2}{21} \le t \le \dfrac{3}{7}. \end{cases}$$

We obtain that

$$P(t)^2 = \begin{cases} \dfrac{1}{14}(2 - 49t^2) & \text{if } 0 \le t \le \dfrac{2}{21}, \\ \left(\dfrac{3}{7} - t\right)^2 & \text{if } \dfrac{2}{21} \le t \le \dfrac{3}{7}. \end{cases}$$

Hence, the S-invariant of E is

$$S(E) = 7\left(\frac{50}{3969} + \frac{1}{81}\right) = \frac{11}{63}.$$

Also, we have that $K_{S_2'} = \psi^*(K_{S^7_{4,2}}) - \frac{2}{7}L_2 - \frac{4}{7}E$, which implies that $A_{S^7_{4,2}}(E) = \frac{3}{7}$ and $\frac{A(E)}{S(E)} = \frac{27}{11} > 1$. Moreover, we have

$$P(t)\cdot E = \begin{cases} \frac{7}{2}t & \text{if } 0 \le t \le \frac{2}{21}, \\ \frac{3}{7} - t & \text{if } \frac{2}{21} \le t \le \frac{3}{7}. \end{cases}$$

For the point $p \in C_2 \cap E$, we have

$$h(t) = \begin{cases} \frac{49}{8}t^2 & \text{if } 0 \le t \le \frac{2}{21}, \\ \left(\frac{3}{7} - t\right)\left(-\frac{2}{7} + 3t\right) + \frac{1}{2}\left(\frac{3}{7} - t\right)^2 & \text{if } \frac{2}{21} \le t \le \frac{3}{7}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 14\left(\frac{1}{567} + \frac{2}{81}\right) = \frac{10}{27}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^7_{4,2}) > 1$.

For the point $p \in L_2 \cap E$, we have

$$h(t) = \begin{cases} \frac{7}{4}t^2 + \frac{49}{8}t^2 & \text{if } 0 \le t \le \frac{2}{21}, \\ \left(\frac{3}{7} - t\right)\left(-\frac{1}{7} + 2t\right) + \frac{1}{2}\left(\frac{3}{7} - t\right)^2 & \text{if } \frac{2}{21} \le t \le \frac{3}{7}. \end{cases}$$

Then $S(W^E_{\bullet,\bullet}; p) = 14\left(\frac{1}{441} + \frac{4}{189}\right) = \frac{62}{189}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^7_{4,2}) > 1$.

For $p \in E \setminus (C_2 \cup L_2)$, we have

$$h(t) = \begin{cases} \frac{49}{8}t^2 & \text{if } 0 \le t \le \frac{2}{21}, \\ \frac{1}{2}\left(\frac{3}{7} - t\right)^2 & \text{if } \frac{2}{21} \le t \le \frac{3}{7}. \end{cases}$$

Then $S(W^E_{\bullet,\bullet}; p) = 14\left(\frac{1}{567} + \frac{1}{162}\right) = \frac{1}{9}$. Since the point p is smooth, the log discrepancy $A_{L_1}(p)$ is equal to 1, which implies that $\delta_p(S^7_{4,2}) > 1$. □

Lemma 3.9. *Let $p \in S^7_{4,2}$ be a smooth point. Then we have $\delta_p(S^7_{4,2}) > 1$.*

Proof. We set $\lambda := \frac{3}{4}$ for simplicity. Let $D_S \equiv -K_{S^7_{4,2}}$ be an effective $\mathbb{Q}$-divisor. We have

$$K_{S_1'} = \varphi^*(K_{S^7_{4,2}}) - \frac{2}{7}L_1 \text{ and } \varphi^*(D_S) = D_1 + \alpha L_1,$$

where D_1 is the strict transform of D_S and α is the multiplicity of D_S at p. Suppose that the pair $(S^7_{4,2}, \lambda D_S)$ is not log canonical at the point p. Then the pair $(S_1', \lambda D_1 + (\lambda\alpha + \frac{2}{7})L_1)$ is not log canonical at a point $p_1 \in S_1'$ such that $p_1 \notin L_1$ and $\varphi(p_1) = p$. This implies that the pair $(S', \lambda D + (\lambda\alpha + \frac{2}{7})L)$ is not log canonical at the point $q = \pi_1(p_1)$ on S', where D and L are the pushforwards of D_1 and L_1, respectively. Note that by a weighted blow-up with weights $(1, 4)$ at the singular point of S in Figure 4, we obtain the birational morphism $f\colon S' \to S$. Moreover, we have

$$K_{S'} = f^*(K_S) + \frac{2}{3}E,$$

where E is the exceptional curve with $E^2 = -\frac{3}{4}$. Note also that $D + (\alpha + \frac{2}{7})L \equiv -K_{S'}$. Hence, the pair $(S, \lambda(D_2 + (\alpha + \frac{2}{7})L_2))$ is not log canonical at the point $f(q)$, where D_2 and L_2 are the pushforwards of D and L, respectively. This is a contradiction to Lemma 3.5. Hence, we obtain $\delta_p(S^7_{4,2}) > 1$. $\square$

Theorem 3.10. *The surface $S^7_{4,2}$ is K-stable.*

Proof. By Lemmas 3.8 and 3.9, we see that the surface $S^7_{4,2}$ is K-stable. $\square$

Next, we deal with the surfaces $S^{n+2}_{n,m}$. Let $\pi_1\colon S^{(2)}_1 \to \mathbb{P}(1,1,n)$ be the surface obtained by blowing-up m distinct points $p_1, \dots, p_m$ on a curve $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,n)}(1)|$, and smooth general points $q_1, \dots, q_{n+2}$. For any $n+1$ points $q_1, \dots, \hat{q}_i, \dots, q_{n+2}$, i.e., omitting q_i, there exists the unique irreducible curve $c_i \in |\mathcal{O}_{P(1,1,n)}(n)|$ that passes through these points. Moreover, we have the following intersection numbers:

$$\ell^2 = \frac{1}{n}, \quad \ell \cdot c_i = 1, \quad c_i^2 = n \text{ and } c_i \cdot c_j = n$$

for all $1 \le i, j \le n+2$.

By contracting the strict transform L of ℓ along π_1, we obtain the birational morphism $\varphi\colon S^{(2)}_1 \to S^{n+2}_{n,m}$. Let $\pi_2\colon S^{(2)}_2 \to S^{(2)}_1$ be the weighted blow-up of $S^{(2)}_1$ at the singular point $q \in L$ with weights $(1,1)$. These birational morphisms can be illustrated by the following diagram.

$$\begin{array}{ccc} & S^{(2)}_2 & \\ & \downarrow{\scriptstyle \pi_2} & \\ & S^{(2)}_1 & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \varphi} \\ \mathbb{P}(1,1,n) & & S^{n+2}_{n,m} \end{array}$$

FIGURE 6

On the surface $S^{(2)}_1$, let L and C_i be the strict transforms of ℓ and c_i, respectively. Also, let E_j be the exceptional curves over each point p_j for $1 \le j \le m$. Then we have the following intersection numbers:

$$L^2 = \frac{1}{n} - m, \quad L \cdot C_i = 1, \quad C_i^2 = -1, \quad C_i \cdot C_j = 0 \text{ for } i \neq j, \quad E_j^2 = -1,$$
$$L \cdot E_j = 1, \quad E_i \cdot E_j = 0 \text{ for } i \neq j \text{ and } C_i \cdot E_j = 0 \text{ for all } i, j.$$

Lemma 3.11. *For a smooth point $p \in L$, we have*

$$\delta_p(S_{n,m}^{n+2}) \geq \min\left\{\frac{3(n+1)^2}{mn+2n^2+4n+1}, \frac{3n}{n+1}\right\}.$$

Proof. We have

$$-K_{S_1^{(2)}} \equiv \frac{n+2}{n+1}L + \frac{1}{n+1}(C_1+\cdots+C_{n+2}) + \frac{1}{n+1}(E_1+\cdots+E_m),$$

and

$$K_{S_1^{(2)}} = \varphi^*(K_{S_{n,m}^{n+2}}) + \left(-1+\frac{n+1}{mn-1}\right)L.$$

Hence, $A_{S_{n,m}^{n+2}}(L) = \frac{n+1}{mn-1}$, and we obtain that

$$\varphi^*(-K_{S_{n,m}^{n+2}}) - tL \equiv \left(\frac{1}{n+1}+\frac{n+1}{mn-1}-t\right)L + \frac{1}{n+1}(C_1+\cdots+C_{n+2}+E_1+\cdots+E_m).$$

Since the intersection matrix of $C_1,\ldots,C_{n+2},E_1,\ldots,E_m$ is negative definite, the pseudoeffective threshold of L is $\frac{1}{n+1}+\frac{n+1}{mn-1}$.

Moreover, the positive part $P(t)$ and negative part $N(t)$ of the Zariski decomposition of $\varphi^*(-K_{S_{n,m}^{n+2}}) - tL$ are as follows:

$$P(t) = \begin{cases} \varphi^*(-K_{S_{n,m}^{n+2}}) - tL & \text{if } 0 \leq t \leq \dfrac{n+1}{mn-1}, \\ \left(\dfrac{1}{n+1}+\dfrac{n+1}{mn-1}-t\right)(L+C_1+\cdots+C_{n+2}+E_1+\cdots+E_m) & \\ & \text{if } \dfrac{n+1}{mn-1} \leq t \leq \dfrac{1}{n+1}+\dfrac{n+1}{mn-1}, \end{cases}$$

and

$$N(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \dfrac{n+1}{mn-1}, \\ \left(-\dfrac{n+1}{mn-1}+t\right)(C_1+\cdots+C_{n+2}+E_1+\cdots+E_m) & \\ & \text{if } \dfrac{n+1}{mn-1} \leq t \leq \dfrac{1}{n+1}+\dfrac{n+1}{mn-1}. \end{cases}$$

Therefore, we have

$$P(t)^2 = \begin{cases} \dfrac{m+n+2}{mn-1} + \left(\dfrac{1}{n}-m\right)t^2 & \text{if } 0 \leq t \leq \dfrac{n+1}{mn-1}, \\ \dfrac{(n+1)^2}{n}\left(\dfrac{1}{n+1}+\dfrac{n+1}{mn-1}-t\right)^2 & \text{if } \dfrac{n+1}{mn-1} \leq t \leq \dfrac{1}{n+1}+\dfrac{n+1}{mn-1}. \end{cases}$$

Hence, we obtain that

$$S(L) = \frac{mn+2n^2+4n+1}{3(mn-1)(n+1)}.$$

Now, we compute the $S(W^L_{\bullet,\bullet};p)$ invariant for smooth points $p\in L$. If $p\notin L\cap(\cup_i C_i\cup\cup_j E_j)$, then we have

$$h(t)=\begin{cases}\frac{1}{2}\left(\frac{1}{n}-m\right)^2t^2 & \text{if } 0\le t\le \frac{n+1}{mn-1},\\ \frac{1}{2}\left(\frac{1}{n+1}+\frac{n+1}{mn-1}-t\right)^2\left(\frac{(n+1)^2}{n}\right)^2 & \text{if } \frac{n+1}{mn-1}\le t\le \frac{1}{n+1}+\frac{n+1}{mn-1}.\end{cases}$$

If $p\in L\cap(\cup_i C_i\cup\cup_j E_j)$, then we have

$$h(t)=\begin{cases}\frac{1}{2}\left(\frac{1}{n}-m\right)^2t^2 & \text{if } 0\le t\le \frac{n+1}{mn-1},\\ \frac{(n+1)^2}{n}\left(\frac{1}{n}+\frac{n+1}{mn-1}-t\right)\left(-\frac{n+1}{mn-1}+t\right) & \\ \qquad +\frac{1}{2}\left(\frac{1}{n+1}+\frac{n+1}{mn-1}-t\right)^2\left(\frac{(n+1)^2}{n}\right)^2 & \\ \qquad\qquad \text{if } \frac{n+1}{mn-1}\le t\le \frac{1}{n+1}+\frac{n+1}{mn-1}. & \end{cases}$$

Hence, we obtain that

$$\begin{aligned}S(W^L_{\bullet,\bullet};p)&=\frac{2(mn-1)}{m+n+2}\int_0^{\tau(L)}h(t)\mathrm{d}t\\ &=\begin{cases}\frac{2(mn-1)}{m+n+2}\left\{\frac{(n+1)^3}{6n^2(mn-1)}+\frac{n+1}{6n^2}\right\} & \text{if } p\notin L\cap(\cup_iC_i\cup\cup_jE_j),\\ \frac{2(mn-1)}{m+n+2}\left\{\frac{(n+1)^3}{6n^2(mn-1)}+\frac{1}{6n(n+1)}+\frac{n+1}{6n^2}\right\} & \text{if } p\in L\cap(\cup_iC_i\cup\cup_jE_j).\end{cases}\\ &=\begin{cases}\frac{n+1}{3n} & \text{if } p\notin L\cap(\cup_iC_i\cup\cup_jE_j),\\ \frac{n^3+(m+4)n^2+(5+3m)n+1}{3n(n+1)(m+n+2)} & \text{if } p\in L\cap(\cup_iC_i\cup\cup_jE_j).\end{cases}\end{aligned}$$

Note that we have

$$\frac{n+1}{3n}<\frac{n^3+(m+4)n^2+(5+3m)n+1}{3n(n+1)(m+n+2)}$$

for all $m,n\ge 2$.

Therefore, for smooth point $p\in L$, we have

$$\delta_p(S^{n+2}_{n,m})\ge\min\left\{\frac{3(n+1)^2}{mn+2n^2+4n+1},\frac{3n}{n+1}\right\}. \qquad \square$$

Now, let us compute $\delta_q(S^{n+2}_{n,m})$ when $m=2$, where $q\in L$ is the singular point. Let $f=\pi_2\circ\varphi$. Then we have

$$K_{S^{(2)}_2}=\pi_2^*(K_{S^{(2)}_1})+\left(-1+\frac{2}{n}\right)E$$

$$= f^*(K_{S^{n+2}_{n,m}}) + \left(-1 + \frac{n+1}{mn-1}\right)\tilde{L} + \left(-1 + \frac{m+1}{mn-1}\right)E,$$

where the maps π_2 and φ are defined in Figure 6. Therefore, we have $A_{S^{n+2}_{n,m}}(E) = \frac{m+1}{mn-1}$.

When $m = 2$, we only need to consider three cases: $(n, m, k) = (3, 2, 5), (4, 2, 6), (5, 2, 7)$. The surface $S^{n+2}_{n,m}$ can be obtained by blowing-up $n+2$ smooth general points in $\mathbb{P}(1, 1, n)$ and $m = 2$ points on a line $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,n)}(1)|$, respectively, and then contracting the $\left(\frac{1}{n} - m\right)$-curve. Note that in $\mathbb{P}(1, 1, n)_{x,y,z}$, the dimension of the family of curves that pass through the $n + 2$ points with multiplicity 3 and m points with multiplicity 1 is $6(n + 2) + m = 6n + 14$. The elements of the linear system $|\mathcal{O}_{\mathbb{P}(1,1,n)}(3n + 4)|$ are of the form

$$f_4(x, y)z^3 + f_{n+4}(x, y)z^2 + f_{2n+4}(x, y)z + f_{3n+4}(x, y) = 0.$$

The sublinear system in which $f_4(x, y) = 0$ has dimension $(n+5)+(2n+5)+(3n+5)-1 = 6n+14$. Hence, such a curve $c \in |\mathcal{O}_{\mathbb{P}(1,1,n)}(3n + 4)|$ exists. Then we have

$$-K_{S^{(2)}_2} = \frac{4}{3}E + \frac{2}{3}\tilde{L} + \frac{1}{3}\tilde{C},$$

where E is the exceptional curve, $\tilde{L}$ and $\tilde{C}$ are the strict transforms of ℓ and c, respectively. The intersection numbers are as follows:

$$E^2 = -n, \quad \tilde{L}^2 = -2, \quad \tilde{C}^2 = -n - 4, \quad E \cdot \tilde{C} = n + 4, \quad E \cdot \tilde{L} = 1, \text{ and } \tilde{L} \cdot \tilde{C} = 0.$$

If we let $f = \pi_2 \circ \varphi$, then we have

$$f^*(-K_{S^{n+2}_{n,2}}) - tE = \left(\frac{1}{3} + \frac{3}{2n-1} - t\right)E + \left(-\frac{1}{3} + \frac{n+1}{2n-1}\right)\tilde{L} + \frac{1}{3}\tilde{C}. \tag{B}$$

Since the intersection matrix of $\tilde{L}$ and $\tilde{C}$ is negative definite, we have $\tau(E) = \frac{1}{3} + \frac{3}{2n-1}$.

Now, we estimate δ-invariant case by case.

3.2.3. *Case: $S^5_{3,2}$.*

Lemma 3.12. *Let $p \in S^5_{3,2}$ be a singular point of type $\frac{1}{5}(1, 3)$. Then we have $\delta_p(S^5_{3,2}) > 1$.*

Proof. We note that $A_{S^5_{3,2}}(E) = \frac{3}{5}$ and by equation B, we have

$$f^*(-K_{S^5_{3,2}}) - tE = \left(\frac{14}{15} - t\right)E + \frac{7}{15}\tilde{L} + \frac{1}{3}\tilde{C}.$$

The positive and negative parts of the Zariski decomposition of $f^*(-K_{S^5_{3,2}}) - tE$ are as follows:

$$P(t) = \begin{cases} \left(\frac{14}{15} - t\right)\left(E + \frac{1}{2}\tilde{L}\right) + \frac{1}{3}\tilde{C} & \text{if } 0 \le t \le \frac{3}{5}, \\ \left(\frac{14}{15} - t\right)\left(E + \frac{1}{2}\tilde{L} + \tilde{C}\right) & \text{if } \frac{3}{5} \le t \le \frac{14}{15}, \end{cases}$$

and

$$N(t) = \begin{cases} \frac{1}{2}t\tilde{L} & \text{if } 0 \le t \le \frac{3}{5}, \\ \frac{1}{2}t\tilde{L} + \left(t - \frac{3}{5}\right)\tilde{C} & \text{if } \frac{3}{5} \le t \le \frac{14}{15}. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} S_{S^5_{3,2}}(E) &= \frac{5}{7}\int_0^{\frac{3}{5}} -\frac{5}{2}\left(\frac{14}{15}-t\right)^2 - \frac{7}{9} + \frac{14}{3}\left(\frac{14}{15}-t\right)\mathrm{d}t + \frac{5}{7}\int_{\frac{3}{5}}^{\frac{14}{15}} \frac{9}{2}\left(\frac{14}{15}-t\right)^2 \mathrm{d}t \\ &= \frac{5}{7}\left(\frac{33}{50}+\frac{1}{18}\right) = \frac{23}{45}, \end{aligned}$$

and hence, $\dfrac{A_{S^5_{3,2}}(E)}{S_{S^5_{3,2}}(E)} = \dfrac{27}{23}$.

In order to apply the Abban–Zhuang theory, we need to compute the following intersection number

$$P(t)\cdot E = \begin{cases} \frac{5}{2}t & \text{if } 0 \le t \le \frac{3}{5}, \\ \frac{9}{2}\left(\frac{14}{15}-t\right) & \text{if } \frac{3}{5} \le t \le \frac{14}{15}. \end{cases}$$

We note that $\tilde{L}\cap\tilde{C} = \emptyset$. For $p \notin E\cap(\tilde{C}\cup\tilde{L})$, we have

$$h(t) = \begin{cases} \frac{25}{8}t^2 & \text{if } 0 \le t \le \frac{3}{5}, \\ \frac{81}{8}\left(\frac{14}{15}-t\right)^2 & \text{if } \frac{3}{5} \le t \le \frac{14}{15}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet};p) = \frac{10}{7}\left(\frac{9}{40}+\frac{1}{8}\right) = \frac{1}{2}$.

For $p \in E\cap\tilde{L}$, we have

$$h(t) = \begin{cases} \frac{5}{4}t^2 + \frac{25}{8}t^2 & \text{if } 0 \le t \le \frac{3}{5}, \\ \frac{9}{4}t\left(\frac{14}{15}-t\right) + \frac{81}{8}\left(\frac{14}{15}-t\right)^2 & \text{if } \frac{3}{5} \le t \le \frac{14}{15}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet};p) = \frac{10}{7}\left(\frac{63}{200}+\frac{77}{360}\right) = \frac{34}{45}$.

For $p \in E\cap\tilde{C}$, since $(E\cdot\tilde{C})_p \le E\cdot\tilde{C} = 7$, we have

$$\begin{aligned} &h(t) = \frac{25}{8}t^2 \quad \text{if } 0 \le t \le \frac{3}{5}, \\ &h(t) \le \frac{63}{2}\left(\frac{14}{15}-t\right)\left(t-\frac{3}{5}\right) + \frac{81}{8}\left(\frac{14}{15}-t\right)^2 \quad \text{if } \frac{3}{5} \le t \le \frac{14}{15}. \end{aligned}$$

Hence, $S(W^E_{\bullet,\bullet};p) \le \frac{10}{7}\left(\frac{9}{40}+\frac{23}{72}\right) = \frac{7}{9}$, and we have $\delta_p(S^5_{3,2}) > 1$ for $p \in E$.

Therefore, by Lemma 3.11 and the above argument, we have $\delta_p(S^5_{3,2}) > 1$ for the singular point $p \in S^5_{3,2}$. ◻

Lemma 3.13. *Let $p \in S^5_{3,2}$ be a smooth point. Then we have $\delta_p(S^5_{3,2}) > 1$.*

Proof. Let $\psi\colon S^6_{3,2} \to S^5_{3,2}$ be a blow-up at p and $D \equiv -K_{S^5_{3,2}}$. Then we have $K_{S^6_{3,2}} = \psi^*(K_{S^5_{3,2}}) + E$ and $\psi^*(D) = D' + \alpha E$, where D' is the strict transform of D, E is the exceptional curve over the point p and α is the multiplicity of D at p. Let $\lambda \coloneqq \frac{3}{4}$. Suppose that $(S^5_{3,2}, \lambda D)$ is not log canonical at p. Then by Lemma 2.18, we have $\alpha > \frac{4}{3}$. Moreover, the pair $(S^6_{3,2}, \lambda D' + (\lambda\alpha - 1)E)$ is not log canonical at a point $p_1 \in S^6_{3,2}$ such that $p_1 \notin E$ and $\psi(p_1) = p$. On the other hand, since $-K_{S^6_{3,2}} \equiv D' + (\alpha-1)E$, the pair $(S^6_{3,2}, \lambda(D'+(\alpha-1)E))$ is log canonical at p_1 by Lemma 3.5. Since $0 < \lambda\alpha - 1 < \lambda(\alpha-1)$, the pair $(S^6_{3,2}, \lambda D' + (\lambda\alpha-1)E)$ is log canonical at p_1, which is a contradiction. Thus, we obtain that $\alpha_p(S^5_{3,2}) \geq \frac{3}{4}$. Since $\frac{3}{2}\alpha_p(S^5_{3,2}) \leq \delta_p(S^5_{3,2})$ by Theorem 2.16, we have $\delta_p(S^5_{3,2}) > 1$. □

Theorem 3.14. *The surface $S^5_{3,2}$ is K-stable.*

Proof. By Lemmas 3.12 and 3.13, we see that the surface $S^5_{3,2}$ is K-stable. □

3.2.4. *Case: $S^6_{4,2}$.*

Lemma 3.15. *Let $p \in S^6_{4,2}$ be the singular point of type $\frac{1}{7}(1,4)$. Then we have $\delta_p(S^6_{4,2}) > 1$.*

Proof. We note that $A_{S^6_{4,2}}(E) = \frac{3}{7}$ and by equation B, we have

$$f^*(-K_{S^6_{4,2}}) - tE = \left(\frac{16}{21} - t\right)E + \frac{8}{21}\tilde{L} + \frac{1}{3}\tilde{C}.$$

The positive and negative parts of the Zariski decomposition are as follows:

$$P(t) = \begin{cases} \left(\frac{16}{21} - t\right)\left(E + \frac{1}{2}\tilde{L}\right) + \frac{1}{3}\tilde{C} & \text{if } 0 \leq t \leq \frac{3}{7}, \\ \left(\frac{16}{21} - t\right)\left(E + \frac{1}{2}\tilde{L} + \tilde{C}\right) & \text{if } \frac{3}{7} \leq t \leq \frac{16}{21}, \end{cases}$$

and

$$N(t) = \begin{cases} \frac{1}{2}t\tilde{L} & \text{if } 0 \leq t \leq \frac{3}{7}, \\ \frac{1}{2}t\tilde{L} + \left(t - \frac{3}{7}\right)\tilde{C} & \text{if } \frac{3}{7} \leq t \leq \frac{16}{21}. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} S_{S^6_{4,2}}(E) &= \frac{7}{8}\int_0^{\frac{3}{7}} -\frac{7}{2}\left(\frac{16}{21} - t\right)^2 - \frac{8}{9} + \frac{16}{3}\left(\frac{16}{21} - t\right) dt + \frac{7}{8}\int_{\frac{3}{7}}^{\frac{16}{21}} \frac{9}{2}\left(\frac{16}{21} - t\right)^2 dt \\ &= \frac{7}{8}\left(\frac{39}{98} + \frac{1}{18}\right) = \frac{25}{63}, \end{aligned}$$

and $\frac{A_{S^6_{4,2}}(E)}{S_{S^6_{4,2}}(E)} = \frac{27}{25}$. Moreover, we have

$$P(t) \cdot E = \begin{cases} \frac{7}{2}t & \text{if } 0 \leq t \leq \frac{3}{7}, \\ \frac{9}{2}\left(\frac{16}{21} - t\right) & \text{if } \frac{3}{7} \leq t \leq \frac{16}{21}. \end{cases}$$

For $p \notin E \cap (\tilde{C} \cup \tilde{L})$, we have

$$h(t) = \begin{cases} \frac{49}{8}t^2 & \text{if } 0 \leq t \leq \frac{3}{7}, \\ \frac{81}{8}\left(\frac{16}{21} - t\right)^2 & \text{if } \frac{3}{7} \leq t \leq \frac{16}{21}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet};p) = \frac{7}{4}\left(\frac{9}{56}+\frac{1}{8}\right) = \frac{1}{2}$.

For $p \in E \cap \tilde{L}$, we have

$$h(t) = \begin{cases} \dfrac{7}{4}t^2 + \dfrac{49}{8}t^2 & \text{if } 0 \le t \le \dfrac{3}{7}, \\ \dfrac{9}{4}t\left(\dfrac{16}{21}-t\right) + \dfrac{81}{8}\left(\dfrac{16}{21}-t\right)^2 & \text{if } \dfrac{3}{7} \le t \le \dfrac{16}{21}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet};p) = \frac{7}{4}\left(\frac{81}{392}+\frac{97}{504}\right) = \frac{44}{63}$.

For $p \in E \cap \tilde{C}$, since $(E \cdot \tilde{C})_p \le E \cdot \tilde{C} = 8$, we have

$$h(t) = \frac{49}{8}t^2 \quad \text{if } 0 \le t \le \frac{3}{7},$$
$$h(t) \le \frac{72}{2}\left(\frac{16}{21}-t\right)\left(t-\frac{3}{7}\right) + \frac{81}{8}\left(\frac{16}{21}-t\right)^2 \quad \text{if } \frac{3}{7} \le t \le \frac{16}{21}.$$

Hence, $S(W^E_{\bullet,\bullet};p) \le \frac{7}{4}\left(\frac{9}{56}+\frac{25}{72}\right) = \frac{8}{9}$, and we have $\delta_p(S^6_{4,2}) > 1$ for $p \in E$.

Therefore, by Lemma 3.11 and the above argument, we have $\delta_p(S^6_{4,2}) > 1$ for the singular point $p \in S^6_{4,2}$. □

Now we compute $\delta_p(S^6_{4,2})$ for a smooth point p.

Lemma 3.16. *Let $p \in S^6_{4,2}$ be a smooth point. Then we have $\delta_p(S^6_{4,2}) > 1$.*

Proof. Let $\psi\colon S^7_{4,2} \to S^6_{4,2}$ be a blow-up at p and $D \equiv -K_{S^6_{4,2}}$ an effective $\mathbb{Q}$-divisor on S. Then we have $K_{S^7_{4,2}} = \psi^*(K_{S^6_{4,2}}) + E$ and $\psi^*(D) = D' + \alpha E$, where D' is the strict transform of D, E is the exceptional curve over p and α is the multiplicity of D at p. Let $\lambda := \frac{3}{4}$. Suppose that $(S^6_{4,2}, \lambda D_S)$ is not log canonical at p. Then by Lemma 2.18, we have $\alpha > \frac{4}{3}$. Moreover, the pair $(S^7_{4,2}, \lambda D' + (\lambda\alpha - 1)E)$ is not log canonical at a point $p_1 \in S^7_{4,2}$ such that $p_1 \notin E$ and $\psi(p_1) = p$. On the other hand, since $-K_{S^7_{4,2}} \equiv D' + (\alpha-1)E$, by Lemma 3.9, the pair $(S^7_{4,2}, \lambda(D' + (\alpha-1)E))$ is log canonical at p_1. Since $0 < \lambda\alpha - 1 < \lambda(\alpha-1)$, the pair $(S^7_{4,2}, \lambda D' + (\lambda\alpha-1)E)$ is log canonical at p_1, which is a contradiction. Thus, we obtain that $\alpha_p(S^6_{4,2}) \ge \frac{3}{4}$. Since $\frac{3}{2}\alpha_p(S^6_{4,2}) \le \delta_p(S^6_{4,2})$ by Theorem 2.16, we have $\delta_p(S^6_{4,2}) > 1$. □

Theorem 3.17. *The surface $S^6_{4,2}$ is K-stable.*

Proof. By Lemmas 3.15 and 3.16, we see that the surface $S^6_{4,2}$ is K-stable. □

3.2.5. *Case: $S^7_{5,2}$.*

Lemma 3.18. *Let $p \in S^7_{5,2}$ be the singular point of type $\frac{1}{9}(1,5)$. Then we have $\delta_p(S^7_{5,2}) = 1$.*

Proof. We note that $A_{S^7_{5,2}}(E) = \frac{1}{3}$ and by equation B, we have

$$f^*(-K_{S^7_{5,2}}) - tE = \left(\frac{2}{3}-t\right)E + \frac{1}{3}\tilde{L} + \frac{1}{3}\tilde{C}.$$

The positive and negative parts of the Zariski decomposition are as follows:

$$P(t) = \begin{cases} \left(\frac{2}{3} - t\right)\left(E + \frac{1}{2}\tilde{L}\right) + \frac{1}{3}\tilde{C} & \text{if } 0 \le t \le \frac{1}{3}, \\ \left(\frac{2}{3} - t\right)\left(E + \frac{1}{2}\tilde{L} + \tilde{C}\right) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \end{cases}$$

and

$$N(t) = \begin{cases} \frac{1}{2}t\tilde{L} & \text{if } 0 \le t \le \frac{1}{3}, \\ \frac{1}{2}t\tilde{L} + \left(t - \frac{1}{3}\right)\tilde{C} & \text{if } \frac{1}{3} \le t \le \frac{2}{3}. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} S_{S^7_{5,2}}(E) &= \int_0^{\frac{1}{3}} -\frac{9}{2}\left(\frac{2}{3} - t\right)^2 - 1 + 6\left(\frac{2}{3} - t\right) \mathrm{d}t + \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{9}{2}\left(\frac{2}{3} - t\right)^2 \mathrm{d}t \\ &= \left(\frac{5}{18} + \frac{1}{18}\right) = \frac{1}{3}, \end{aligned}$$

and $\frac{A_{S^7_{5,2}}(E)}{S_{S^7_{5,2}}(E)} = 1$. Moreover, we have

$$P(t) \cdot E = \begin{cases} \frac{9}{2}t & \text{if } 0 \le t \le \frac{1}{3}, \\ \frac{9}{2}\left(\frac{2}{3} - t\right) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}. \end{cases}$$

For $p \notin E \cap (\tilde{C} \cup \tilde{L})$, we have

$$h(t) = \begin{cases} \frac{81}{8}t^2 & \text{if } 0 \le t \le \frac{1}{3}, \\ \frac{81}{8}\left(\frac{2}{3} - t\right)^2 & \text{if } \frac{1}{3} \le t \le \frac{2}{3}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 2\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{2}$.

For $p \in E \cap \tilde{L}$, we have

$$h(t) = \begin{cases} \frac{9}{4}t^2 + \frac{81}{8}t^2 & \text{if } 0 \le t \le \frac{1}{3}, \\ \frac{9}{4}t\left(\frac{2}{3} - t\right) + \frac{81}{8}\left(\frac{2}{3} - t\right)^2 & \text{if } \frac{1}{3} \le t \le \frac{2}{3}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 2\left(\frac{11}{72} + \frac{13}{72}\right) = \frac{2}{3}$.

For $p \in E \cap \tilde{C}$, since $(E \cdot \tilde{C})_p \le E \cdot \tilde{C} = 9$, we have

$$\begin{aligned} h(t) &= \frac{81}{8}t^2 \quad \text{if } 0 \le t \le \frac{1}{3}, \\ h(t) &\le \frac{81}{2}\left(\frac{2}{3} - t\right)\left(t - \frac{1}{3}\right) + \frac{81}{8}\left(\frac{2}{3} - t\right)^2 \quad \text{if } \frac{1}{3} \le t \le \frac{2}{3}. \end{aligned}$$

Hence, $S(W^E_{\bullet,\bullet};p) \leq 2\left(\frac{1}{8}+\frac{3}{8}\right) = 1$.

Therefore, we have $\delta_p(S^7_{5,2}) = 1$ for the singular point $p \in S^7_{5,2}$. ☐

Lemma 3.19. *Let $p \in S^7_{5,2}$ be a smooth point. Then we have $\delta_p(S^7_{5,2}) > 1$.*

Proof. The proof is exactly the same as the proof of Lemmas 3.5 and 3.13. ☐

We claim that the surface $S^7_{5,2}$ is strictly K-semistable by showing that the automorphism group of $S^7_{5,2}$ is finite.

Lemma 3.20. *The automorphism group* $\mathrm{Aut}(S^7_{5,2})$ *is finite.*

Proof. Consider the natural homomorphism

$$\rho\colon \mathrm{Aut}(S^7_{5,2}) \to \mathbf{O}(K^{\perp}_{S^7_{5,2}})$$

defined by $\sigma \mapsto \sigma^*$, where $\mathbf{O}(K^{\perp}_{S^7_{5,2}})$ is the orthogonal complement to $\mathbb{Z}K_{S^7_{5,2}}$ in $\mathrm{Pic}(S^7_{5,2})$ (see [20, Corollary 8.2.39]). Since the kernel of ρ preserves all geometric basis of $\mathrm{Pic}(S^7_{5,2})$, every automorphism contained in the kernel of ρ descends to an automorphism of $S^0_{5,2}$ which fixes 7 points in general position. Thus, the kernel of ρ is trivial. Furthermore, since $\mathbf{O}(K^{\perp}_{S^7_{5,2}})$ is finite, $\mathrm{Aut}(S^7_{5,2})$ is finite. ☐

Theorem 3.21. *The surface $S^7_{5,2}$ is strictly K-semistable.*

Proof. By Lemmas 3.18, 3.19, 3.20 and Theorem 2.15, we see that the surface $S^7_{5,2}$ is strictly K-semistable. ☐

When $m = 3$, we only have two possibilities: $(n, m) = (3, 3), (4, 3)$. In $\mathbb{P}(1, 1, n)_{x,y,z}$, the dimension of the family of curves that pass through the $n + 2$ points with multiplicity 4 and $m = 3$ points with multiplicity 1 is $10(n + 2) + m = 10n + 23$. The elements of the linear system $|\mathcal{O}_{\mathbb{P}(1,1,n)}(4n + 5)|$ are of the form

$$f_5(x, y)z^4 + f_{n+5}(x, y)z^3 + f_{2n+5}(x, y)z^2 + f_{3n+5}(x, y)z + f_{4n+5}(x, y) = 0.$$

The sublinear system for which $f_5(x, y) = 0$ has dimension $(n+6)+(2n+6)+(3n+6)+(4n+6)-1 = 10n + 23$. Hence, there exists such a curve $c \in |\mathcal{O}_{\mathbb{P}(1,1,n)}(4n + 5)|$. Then we have

$$-K_{S^{(2)}_2} = \frac{5}{4}E + \frac{3}{4}\tilde{L} + \frac{1}{4}\tilde{C},$$

where E is the exceptional curve, $\tilde{L}$ and $\tilde{C}$ are the strict transforms of ℓ and c, respectively. The intersection numbers are as follows:

$$E^2 = -n, \quad \tilde{L}^2 = -3, \quad \tilde{C}^2 = -n - 5, \quad E \cdot \tilde{C} = n + 5, \quad E \cdot \tilde{L} = 1 \text{ and } \tilde{L} \cdot \tilde{C} = 0.$$

Moreover, we have

$$\text{(C)} \qquad \varphi^*(-K_{S^{n+2}_{n,3}}) - tE = \left(\frac{1}{4} + \frac{4}{3n-1} - t\right)E + \left(-\frac{1}{4} + \frac{n+1}{3n-1}\right)\tilde{L} + \frac{1}{4}\tilde{C},$$

where the maps are defined in Figure 6. Since the intersection matrix of $\tilde{L}$ and $\tilde{C}$ is negative definite, we have $\tau(E) = \frac{1}{4} + \frac{4}{3n-1}$.

3.2.6. *Case:* $S^5_{3,3}$.

Lemma 3.22. *Let $p \in S^5_{3,3}$ be the singular point of type $\frac{1}{8}(1,3)$. Then we have $\delta_p(S^5_{3,3}) > 1$.*

Proof. We note that $A_{S^5_{3,3}}(E) = \frac{1}{2}$ and by equation C, we have

$$\varphi^*(-K_{S^5_{3,3}}) - tE = \left(\frac{3}{4} - t\right) E + \frac{1}{4}\tilde{L} + \frac{1}{4}\tilde{C}.$$

The positive and negative parts of the Zariski decomposition are as follows:

$$P(t) = \begin{cases} \left(\frac{3}{4} - t\right)\left(E + \frac{1}{3}\tilde{L}\right) + \frac{1}{4}\tilde{C} & \text{if } 0 \le t \le \frac{1}{2}, \\ \left(\frac{3}{4} - t\right)\left(E + \frac{1}{3}\tilde{L} + \tilde{C}\right) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \end{cases}$$

and

$$N(t) = \begin{cases} \frac{1}{3}t\tilde{L} & \text{if } 0 \le t \le \frac{1}{2}, \\ \frac{1}{3}t\tilde{L} + \left(t - \frac{1}{2}\right)\tilde{C} & \text{if } \frac{1}{2} \le t \le \frac{3}{4}. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} S_{S^5_{3,3}}(E) &= \int_0^{\frac{1}{2}} -\frac{8}{3}\left(\frac{3}{4} - t\right)^2 - \frac{1}{2} + 4\left(\frac{3}{4} - t\right) \mathrm{d}t + \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{16}{3}\left(\frac{3}{4} - t\right)^2 \mathrm{d}t \\ &= \left(\frac{7}{18} + \frac{1}{36}\right) = \frac{5}{12}, \end{aligned}$$

and $\frac{A_{S^5_{3,3}}(E)}{S_{S^5_{3,3}}(E)} = \frac{6}{5}$. Moreover, we have

$$P(t) \cdot E = \begin{cases} \frac{8}{3}t & \text{if } 0 \le t \le \frac{1}{2}, \\ \frac{16}{3}\left(\frac{3}{4} - t\right) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}. \end{cases}$$

For $p \notin E \cap (\tilde{C} \cup \tilde{L})$, we have

$$h(t) = \begin{cases} \frac{32}{9}t^2 & \text{if } 0 \le t \le \frac{1}{2}, \\ \frac{128}{9}\left(\frac{3}{4} - t\right)^2 & \text{if } \frac{1}{2} \le t \le \frac{3}{4}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 2\left(\frac{4}{27} + \frac{2}{27}\right) = \frac{4}{9}$.

For $p \in E \cap \tilde{L}$, we have

$$h(t) = \begin{cases} \frac{8}{9}t^2 + \frac{32}{9}t^2 & \text{if } 0 \le t \le \frac{1}{2}, \\ \frac{16}{9}t\left(\frac{3}{4} - t\right) + \frac{128}{9}\left(\frac{3}{4} - t\right)^2 & \text{if } \frac{1}{2} \le t \le \frac{3}{4}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = 2\left(\frac{5}{27} + \frac{23}{216}\right) = \frac{7}{12}$.

For $p \in E \cap \tilde{C}$, since $(E \cdot \tilde{C})_p \leq E \cdot \tilde{C} = 8$, we have

$$h(t) = \frac{32}{9}t^2 \quad \text{if } 0 \leq t \leq \frac{1}{2},$$
$$h(t) \leq \frac{128}{3}\left(\frac{3}{4} - t\right)\left(t - \frac{1}{2}\right) + \frac{128}{9}\left(\frac{3}{4} - t\right)^2 \quad \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}.$$

Hence, $S(W^E_{\bullet,\bullet}; p) \leq 2\left(\frac{4}{27} + \frac{5}{27}\right) = \frac{2}{3}$. Therefore, by Lemma 3.11 and the above argument, we have $\delta_p(S^5_{3,3}) > 1$ for the singular point $p \in S^5_{3,3}$. □

Now, we show that $\delta_p(S^5_{3,3}) > 1$ for smooth points p.

Lemma 3.23. *Let $p \in S^5_{3,3}$ be a smooth point. Then we have $\delta_p(S^5_{3,3}) > 1$.*

Proof. Let $\pi\colon S^{(3)} \to \mathbb{P}(1,1,3)$ be a blow-up of $\mathbb{P}(1,1,3)$ at six smooth general points $p_1, \ldots, p_6$. Let L be the strict transform of the curve $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,3)}(1)|$ passing through the point p_1. Let $\pi_1\colon S^{(3)}_1 \to S^{(3)}$ be a blow-up at two distinct points $q_1, q_2 \in L$ such that $\pi_1(q_1) \neq p_1$ and $\pi_1(q_2) \neq p_1$. Then by contracting the strict transform L_1 of L, we obtain the birational morphism $\varphi\colon S^{(3)}_1 \to S$, where S is the surface in Figure 4. For the singular point q in $S^{(3)}_1$ with type $\frac{1}{3}(1,1)$, let $\pi_2\colon S^{(3)}_2 \to S^{(3)}_1$ be the weighted blow-up with weights $(1,1)$. We consider the following diagram.

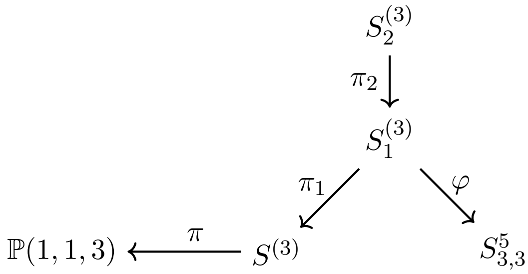

FIGURE 7

We set $\lambda := \frac{3}{4}$. Let $D_S \equiv -K_{S^5_{3,3}}$ be an effective $\mathbb{Q}$-divisor on $S^5_{3,3}$. Suppose that the pair $(S^5_{3,3}, \lambda D_S)$ is not log canonical at a smooth point p. We have

$$K_{S^{(3)}_1} = \varphi^*(K_{S^5_{3,3}}) - \frac{1}{2}L_1 \text{ and } \varphi^*(D_S) = D_1 + \alpha L_1,$$

where D_1 is the strict transform of D_S and α is the multiplicity of D_S at p. Then the pair $(S^{(3)}_1, \lambda D_1 + (\lambda\alpha + \frac{1}{2})L_1)$ is not log canonical at a point $p_1 \in S^{(3)}_1$ such that $p_1 \notin L_1$ and $\varphi(p_1) = p$. This implies that the pair $(S^{(3)}, \lambda D + (\lambda\alpha + \frac{1}{2})L)$ is not log canonical at the point $q = \pi_1(p_1)$ on $S^{(3)}$, where D and L are the pushforwards of D_1 and L_1, respectively. Then the pair $(S^{(3)}, \lambda D)$ is not log canonical at q. Note that by blowing-up a point $p_7 \notin L_1$, we obtain the birational morphism $\psi\colon S \to S^{(3)}$, where S is a complete intersection of two degree 4 hypersurfaces in $\mathbb{P}(1,1,2,2,3)$ in Figure 4. Note that $D + (\alpha + \frac{1}{2})L \equiv -K_{S^{(3)}}$, and we have

$$K_S = \psi^*(K_{S^{(3)}}) + E, \quad \psi^*\left(D + \left(\alpha + \frac{1}{2}\right)L\right) = D_2 + \left(\alpha + \frac{1}{2}\right)L_2 + \beta E,$$

where D_2 and L_2 are the pushforwards of D and L, respectively, E is the exceptional curve over the point p_7, and β is the multiplicity of $\left(D + \left(\alpha + \frac{1}{2}\right) L\right)$ at the point p_7. Hence, the pair $(S, \lambda(D_2 + (\alpha + \frac{1}{2})L_2) + (\lambda\beta - 1)E)$ is not log canonical at $q' \in D_2$. On the other hand, by Lemma 3.5, the pair $(S, \lambda(D_2 + (\alpha + \frac{1}{2})L_2 + (\beta - 1)E))$ is log canonical at a point q'. Since $0 < \lambda\beta - 1 < \lambda(\beta - 1)$, the pair $(S, \lambda(D_2 + (\alpha + \frac{1}{2})L_2) + (\lambda\beta - 1)E)$ is log canonical at $q' \in D_2$. This is a contradiction, and we obtain $\delta_p(S^5_{3,3}) > 1$. ☐

Theorem 3.24. *The surface $S^5_{3,3}$ is K-stable.*

Proof. By Lemmas 3.22 and 3.23, we see that the surface $S^5_{3,3}$ is K-stable. ☐

3.2.7. *Case: $S^6_{4,3}$.*

Lemma 3.25. *Let $p \in S^6_{4,3}$ be the singular point of type $\frac{1}{11}(1, 4)$. Then we have $\delta_p(S^6_{4,3}) > 1$.*

Proof. We note that $A_{S^6_{4,3}}(E) = \frac{4}{11}$ and by equation C, we have

$$\varphi^*(-K_{S^6_{4,3}}) - tE = \left(\frac{27}{44} - t\right) E + \frac{9}{44}\tilde{L} + \frac{1}{4}\tilde{C}.$$

The positive and negative parts of the Zariski decomposition are as follows:

$$P(t) = \begin{cases} \left(\frac{27}{44} - t\right)\left(E + \frac{1}{3}\tilde{L}\right) + \frac{1}{4}\tilde{C} & \text{if } 0 \le t \le \frac{4}{11}, \\ \left(\frac{27}{44} - t\right)\left(E + \frac{1}{3}\tilde{L} + \tilde{C}\right) & \text{if } \frac{4}{11} \le t \le \frac{27}{44}, \end{cases}$$

and

$$N(t) = \begin{cases} \frac{1}{3}t\tilde{L} & \text{if } 0 \le t \le \frac{4}{11}, \\ \frac{1}{3}t\tilde{L} + \left(t - \frac{4}{11}\right)\tilde{C} & \text{if } \frac{4}{11} \le t \le \frac{27}{44}. \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} S_{S^6_{4,3}}(E) &= \frac{11}{9}\int_0^{\frac{4}{11}} -\frac{11}{3}\left(\frac{27}{44} - t\right)^2 - \frac{9}{16} + \frac{9}{2}\left(\frac{27}{44} - t\right) \mathrm{d}t + \frac{11}{9}\int_{\frac{4}{11}}^{\frac{27}{44}} \frac{16}{3}\left(\frac{27}{44} - t\right)^2 \mathrm{d}t \\ &= \frac{11}{9}\left(\frac{260}{1089} + \frac{1}{36}\right) = \frac{43}{132}, \end{aligned}$$

and $\frac{A_{S^6_{4,3}}(E)}{S_{S^6_{4,3}}(E)} = \frac{48}{43}$. Moreover, we have

$$P(t) \cdot E = \begin{cases} \frac{11}{3}t & \text{if } 0 \le t \le \frac{4}{11}, \\ \frac{16}{3}\left(\frac{27}{44} - t\right) & \text{if } \frac{4}{11} \le t \le \frac{27}{44}. \end{cases}$$

For $p \notin E \cap (\tilde{C} \cup \tilde{L})$, we have

$$h(t) = \begin{cases} \dfrac{121}{18} t^2 & \text{if } 0 \le t \le \dfrac{4}{11}, \\ \dfrac{128}{9} \left(\dfrac{27}{44} - t \right)^2 & \text{if } \dfrac{4}{11} \le t \le \dfrac{27}{44}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = \frac{22}{9} \left(\frac{32}{297} + \frac{2}{27} \right) = \frac{4}{9}$.

For $p \in E \cap \tilde{L}$, we have

$$h(t) = \begin{cases} \dfrac{11}{9} t^2 + \dfrac{121}{18} t^2 & \text{if } 0 \le t \le \dfrac{4}{11}, \\ \dfrac{16}{9} t \left(\dfrac{27}{44} - t \right) + \dfrac{128}{9} \left(\dfrac{27}{44} - t \right)^2 & \text{if } \dfrac{4}{11} \le t \le \dfrac{27}{44}. \end{cases}$$

Hence, $S(W^E_{\bullet,\bullet}; p) = \frac{22}{9} \left(\frac{416}{3267} + \frac{235}{2376} \right) = \frac{73}{132}$.

For $p \in E \cap \tilde{C}$, since $(E \cdot \tilde{C})_p \le E \cdot \tilde{C} = 9$, we have

$$h(t) = \frac{121}{18} t^2 \quad \text{if } 0 \le t \le \frac{4}{11},$$
$$h(t) \le \frac{144}{3} \left(\frac{27}{44} - t \right) \left(t - \frac{4}{11} \right) + \frac{128}{9} \left(\frac{27}{44} - t \right)^2 \quad \text{if } \frac{4}{11} \le t \le \frac{27}{44}.$$

Hence, $S(W^E_{\bullet,\bullet}; p) \le \frac{22}{9} \left(\frac{32}{297} + \frac{43}{216} \right) = \frac{3}{4}$. Therefore, by Lemma 3.11 and the above argument, we have $\delta_p(S^6_{4,3}) > 1$ for the singular point $p \in S^6_{4,3}$. □

Lemma 3.26. *Let $p \in S^6_{4,3}$ be a smooth point. Then we have $\delta_p(S^6_{4,3}) > 1$.*

Proof. Let $\pi\colon S^{(4)} \to \mathbb{P}(1,1,4)$ be a blow-up of $\mathbb{P}(1,1,4)$ at seven smooth general points $p_1, \ldots, p_7$. Let L be the strict transform of the curve $\ell \in |\mathcal{O}_{\mathbb{P}(1,1,4)}(1)|$ passing through the point p_1. Let $\pi_1\colon S^{(4)}_1 \to S^{(4)}$ be a blow-up at two distinct points $q_1, q_2 \in L$ such that $\pi_1(q_1) \ne p_1$ and $\pi_1(q_2) \ne p_1$. Then by contracting the strict transform L_1 of L, we obtain the birational morphism $\varphi\colon S^{(4)}_1 \to S'$, where S' is the surface in Figure 5. For the singular point q in $S^{(4)}_1$ with type $\frac{1}{3}(1,1)$, let $\pi_2\colon S^{(4)}_2 \to S^{(4)}_1$ be the weighted blow-up with weights $(1,1)$. We consider the following diagram.

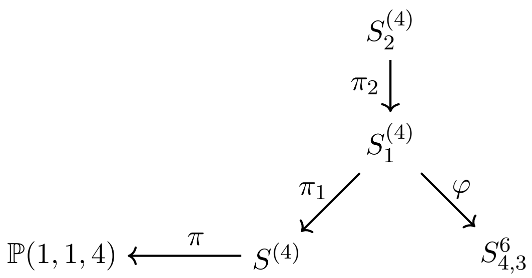

FIGURE 8

We set $\lambda := \frac{3}{4}$. Let $D_S \equiv -K_{S^6_{4,3}}$ be an effective $\mathbb{Q}$-divisor on $S^6_{4,3}$. Suppose that the pair $(S^6_{4,3}, \lambda D_S)$ is not log canonical at a smooth point p. We have

$$K_{S_1^{(4)}} = \varphi^*(K_{S^6_{4,3}}) - \frac{6}{11}L_1 \text{ and } \varphi^*(D_S) = D_1 + \alpha L_1,$$

where D_1 is the strict transform of D_S and α is the multiplicity of D_S at p. Then the pair $(S_1^{(4)}, \lambda D_1 + (\lambda\alpha + \frac{6}{11})L_1)$ is not log canonical at a point $p_1 \in S_1^{(4)}$ such that $p_1 \notin L_1$ and $\varphi(p_1) = p$. This implies that the pair $(S^{(4)}, \lambda D + (\lambda\alpha + \frac{6}{11})L)$ is not log canonical at the point $q = \pi_1(p_1)$ on $S^{(4)}$, where D and L are the pushforwards of D_1 and L_1, respectively. Then the pair $(S^{(4)}, \lambda D)$ is not log canonical at q. Note that by blowing-up at a point $p_8 \notin L_1$, we obtain the birational morphism $\psi\colon S' \to S^{(4)}$, where S' is a degree 6 hypersurface in $\mathbb{P}(1,1,2,3)$ in Figure 5. Note that $D + (\alpha + \frac{6}{11})L \equiv -K_{S^{(4)}}$, and we have

$$K_{S'} = \psi^*(K_{S^{(4)}}) + E, \quad \psi^*\left(D + \left(\alpha + \frac{6}{11}\right)L\right) = D_2 + \left(\alpha + \frac{6}{11}\right)L_2 + \beta E,$$

where D_2 and L_2 are the strict transforms of D and L, respectively, E is the exceptional curve over the point p_8, and β is the multiplicity of $\left(D + \left(\alpha + \frac{6}{11}\right)L\right)$ at the point p_8. Hence, the pair $(S', \lambda(D_2 + (\alpha + \frac{6}{11})L_2) + (\lambda\beta - 1)E)$ is not log canonical at $q' \in D_2$. On the other hand, by Lemma 3.9, the pair $(S', \lambda(D_2 + (\alpha + \frac{6}{11})L_2 + (\beta - 1)E))$ is log canonical at a point q. Since $0 < \lambda\beta - 1 < \lambda(\beta - 1)$, the pair $(S', \lambda(D_2 + (\alpha + \frac{6}{11})L_2) + (\lambda\beta - 1)E)$ is log canonical at q'. This is a contradiction, and we obtain $\delta_p(S^6_{4,3}) > 1$. □

Theorem 3.27. *The surface $S^6_{4,3}$ is K-stable.*

Proof. By Lemmas 3.25 and 3.26, we see that the surface $S^6_{4,3}$ is K-stable. □

Finally, we prove the main result of this paper.

Proof of Theorem 1.4. By Remark 3.3, Theorems 3.2, 3.14, 3.7, 3.24, 3.17, 3.10, 3.27 and 3.21, we complete the proof. □

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