Convex Matrix Inequalities Versus Linear Matrix Inequalities

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Convex Matrix Inequalities Versus Linear Matrix Inequalities

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***Abstract*—Most linear control problems lead directly to matrix inequalities (MIs). Many of these are badly behaved but a classical core of problems are expressible as linear matrix inequalities (LMIs). In many engineering systems problems convexity has all of the advantages of a LMI. Since LMIs have a structure which is seemingly much more rigid than convex MIs, there is the hope that a convexity based theory will be less restrictive than LMIs.** *How much more restrictive are LMIs than convex MIs?*

There are two fundamentally different classes of linear systems problems: dimension free and dimension dependent. A dimension free MI is a MI where the unknowns are g-tuples of matrices and appear in the formulas in a manner which respects matrix multiplication. Most of the classic MIs of control theory are dimension free. Dimension dependent MIs have unknowns which are tuples of numbers. The two classes behave very differently and this survey describes what is known in each case about the relation between convex MIs and LMIs. The proof techniques involve and necessitate new developments in the field of semialgebraic geometry.

***Index Terms*—Algebraic approaches, convex optimization, linear control systems, linear matrix inequality (LMI).**

I. Introduction

OPTIMIZATION plays, as it has for many years, a major role in applications, especially in the area of control. One of the biggest recent revolutions in optimization, called semidefinite programming, is a methodology for solving linear matrix inequalities (LMIs) proposed in the 1994 book by Nesterov and Nemirovski [31]. See the recent survey [32] for a description of current application areas and methods. Semidefinite programming led to a major advance in control, when it was realized in the 1990's that most linear control problems are matrix inequalities, abbreviated MIs. Loosely, an MI is an inequality $F \succeq 0$, where F is a square matrix valued function, the inequality is in the sense of positive semidefinite matrix, and of interest is the feasible set consisting of those points in the domain of F for which the inequality holds. While many of the MIs of linear control are badly behaved, a classical core of these problems can be written as LMIs, meaning, roughly, that F can be chosen affine linear. Thus semidefinite program can be used to solve a variety of problems which were previously intractable.

Systems problems, very directly and by routine methods, produce lists of messy matrix inequalities. A practical issue is then to transform a system problem to a nice MI or prove this is impossible. For the purposes of this survey, we interpret nice MIs as either convex MIs or LMIs, in part because convexity generally guarantees numerical success and also because of the importance of convexity and preponderance of LMIs in the systems literature. Thus, two basic questions are:

Which MIs transform to convex MIs (to LMIs)?

How much more general are convex MIs than LMIs?

A main tool in addressing these problems is semialgebraic geometry **(SAG)**, i.e., the study of systems of polynomial inequalities. The subject includes the famous problem of Hilbert on sums of squares and, along with various "Positivstellensätze", has for nearly a century been a vibrant area of research, c.f [5]. Not only do we need classical SAG but also we shall require a matrix-variable version of SAG currently under development as the main tool (as we see in this article) in studying the relation between LMIs and convex MIs.

There are two fundamentally different classes of linear systems problems. **Dimension dependent** problems—those which depend explicitly on the dimension of the system—lead to traditional semialgebraic geometry. **Dimension free** problems, synonymously, **matrix variable** problems—those which do not depend explicitly on the dimension of the system—lead to a new area which might be called matrix variable semialgebraic geometry, c.f [18]. The classic systems problems found in the book of Zhou-Doyle-Glover [42], and systems problems expressible as LMIs found in the 1997 book of Skelton-Iwasaki-Grigoriadis [41], are of the dimension free matrix variable variety.

This article focuses on the more modest of the two questions above. Namely, *which convex MIs are really LMIs in disguise?* The survey describes what is known about the relation between convex MIs and LMIs, first in the dimension free (matrix variable) case and then considering two paradigm dimension dependent problems.

The dimension dependent problems ask, for a given convex subset S of $\mathbb{R}^g$:

1) does S have a LMI representation (see Section V)?
2) does S lift to a set with a LMI representation (see Section VI)?

At this juncture, in both the dimension free and dimension dependent cases, there are elegant theorems settling toy problems which make strong suggestions as to the general structures. The results outlined in this survey can be quickly absorbed and hopefully will offer guidance on the bumpy MI road ahead. We hope the reader will enjoy the glimpse here and forgive us for the

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omission of many details and references to a rapidly growing literature.

Before continuing with the introduction, we direct the reader who is not interested in matrix variables (dimension free) to Sections V and VI which deal with the questions of which convex sets in $\mathbb{R}^g$ either have a LMI representation or lift to a set with a LMI representation respectively. These sections can be read with minimal reference to earlier ones.

Of course we hope the reader will also be interested in the matrix variable side of this survey and the remainder of this introduction makes the case that matrix variables are natural, easy to work with (as indeed many readers already do without being aware of it), and necessary from the point of view of writing computer packages to automate standard computations. Sections I-A–I-C compare and contrast dimension free and dimension dependent problems, the latter in relation to computer algebra. The introduction concludes with a brief subsection, Section I-D, outlining the remainder of the paper.

A. Familiar Example

For sake of orientation with respect to matrix variables and our penchant for not fixing a dimension and then considering each entry of each matrix a separate variable, a process we refer to as *disaggregating,* consider the following example.

The most ancient and ubiquitous formula in classical linear control theory is the Riccati inequality

$$AX + XA^T - XBB^TX + C^TC \succeq 0 \tag{1}$$

where "$\succeq$" means positive semidefinite as a matrix. There are three key properties of the Riccati inequality that we wish to isolate. First, the inequality is *dimension free* in that it does not explicitly depend upon the size of the square matrices involved, only that the products are defined.

Secondly, the Riccati polynomial, meaning the left hand side of the inequality (1), is concave in X (concavity/convexity in this context is just as you would imagine and the precise definition is given in Section II-A), since, with $Z := (X+Y)/2$

$$\begin{aligned} 2&(AZ + ZA - ZBB^TZ + C^TC)\\ &-(AX + XA - XBB^TX + C^TC)\\ &-(AY + YA - YBB^TY + C^TC)\\ =&\frac{1}{2}(X-Y)BB^T(X-Y) \succeq 0. \end{aligned} \tag{2}$$

In particular, for given A, B, C, the feasible set of the Riccati inequality is convex in X.

Lastly, the Riccati inequality can be expressed in a form that looks much like a linear matrix inequality. Indeed

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{3}$$

is equivalent to the inequality (1)[1] in that for given matrices A, B, C the inequalities (1) and (3) have the same feasible X. Note that convexity of the feasible set of the Riccati in X also follows from the fact that the left hand side of (3) is affine linear in X.

[1]To see this note that (3) is equivalent to (1) plus

$$AX + XA^T + C^TC \succeq 0. \tag{4}$$

However, (1) implies (4), so (3) is equivalent to (1).

B. To Commute or Not Commute: "Dimension Free" Formulas

In this subsection we illustrate, using the Riccati inequality (3), the distinction between the dimension free noncommutative viewpoint and the dimension dependent (commutative) viewpoint.

As an expression in A, B, C and X, (3) has the same form regardless of the dimension of the system and the sizes of the matrices. In other words, as long as the matrices A, B, C and X have compatible dimension the inequality (3) is meaningful and substantive and its form does not change. This is the matrix variable view of the inequality.

The dimension dependent version of the inequality arises from disaggregating matrices and requires specifying dimensions of A, B, C and X and expressing the inequality as the LMI

$$L_0 + \sum_{j=1}^{g} L_j s_j \succeq 0 \tag{5}$$

where L_j are $d \times d$ matrices (for some d) determined by the given A, B, C and the s_j are unknowns coming from the unknown matrix X. For example, given $A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{2\times 1}, C \in \mathbb{R}^{1\times 2}$, and $X^T = X \in \mathbb{R}^{2\times 2}$, and writing, $X = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$, the choices

$$L_0 := \begin{pmatrix} C^TC & 0 \\ 0 & I \end{pmatrix}, \quad L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_1 \\ a_{21} & 0 & 0 \\ b_1 & 0 & 0 \end{pmatrix}$$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11}+a_{22} & b_2 \\ a_{22}+a_{11} & 2a_{21} & b_1 \\ b_2 & b_1 & 0 \end{pmatrix}$$

$$L_3 := \begin{pmatrix} 0 & a_{12} & 0 \\ a_{12} & 2a_{22} & 0 \\ 0 & b_2 & 0 \end{pmatrix}$$

in inequality (5) produce the inequality (3) (equivalently the inequality (1)).

The situation for $A \in \mathbb{R}^{3\times 3}, B \in \mathbb{R}^{3\times 2}, C \in \mathbb{R}^{2\times 3}, X \in \mathbb{R}^{3\times 3}$ is too horrible to contemplate. The point is that the Riccati inequality expressed in the form of inequality (5), with numbers, rather than matrices, as unknowns, does not scale simply with dimension of the matrices or with the system, whereas the inequality (3) does.

We see problems as naturally falling into one of the two types, **dimension free**, the dimension of the system does not directly enter the statement of the problem, or **dimension dependent**. Most classical systems problems are dimension free, e.g., the H^2 control problem, H^∞ control problem, state estimation problems, etc.. It is an empirical observation that dimension free problems convert to matrix inequalities, while those which are dimension dependent lose this structure. For example, the H^2 control problem converts to solving one Riccati inequality, while the H^∞ control problem converts to solving two Riccatis and a coupling inequality in matrix variables.

In mathematical lexicon the matrix variables in dimension free problems are **noncommuting variables** and from here on we use the terms *dimension free, matrix variable,* and *noncommutative (NC)* interchangeably. Thus, inequality (3) is noncommutative, whereas inequality (5) is commutative (in the commuting variables s_j).

C. Noncommutative Algebra and Matrix Convexity

As an example of a noncommutative function, it is natural to regard the left hand side of inequality (1) as a polynomial

$$r(a,b,c,x) := ax + xa^T - xbb^Tx + c^Tc$$

in the noncommuting variables $a, b, c,$ and x. Substituting square matrices A, B, C, X (with X symmetric) for the variables a, b, c, x produces the left hand side of the Riccati inequality (1). Since there are no relations between variables $a, b, c,$ and x, they are referred to as "free variables." Noncommutative algebra in free variables goes back several decades and is well developed, see, e.g., [8], [9]. This is fortunate, since it is essential to the theory behind results to be described in this paper; more on this is in Section II-A. What is required for analyzing convexity is a theory of noncommutative inequalities, a new area, much of which is pointed to in this survey article.

As an illustration, consider the very simple example of the polynomial in the single (noncommutative) variable x

$$p(x) = x^4$$

and the 2×2 matrices

$$X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Straightforward computation reveals

$$\frac{1}{2}(p(X) + p(Y)) - p\left(\frac{X+Y}{2}\right) = \frac{1}{4}\begin{pmatrix} 41 & 30 \\ 30 & 21 \end{pmatrix} \not\succeq 0$$

which says that $p(x) = x^4$ is *not* convex as a polynomial in the single noncommutative variable x. The freedom to choose a Y which does not commute with X means that, even in the single variable case, the commutative and noncommutative worlds diverge.

Notice that we distinguish between a and x as variables and A and X as matrices. This may seem pedantic but it is essential to put ourselves in the proper mathematical framework, namely that of noncommutative algebra in free variables alluded to above. This allows us to use a body of existing results. It also ties in naturally with computer algebra applications that we shall discuss next.

1) Computer Algebra: While the noncommutative algebra we shall describe in this article underlies analysis of convex vs linear MIs, it also sits at the foundation for computer algebra designed to manipulate matrices as a whole (without resorting to expressions explicitly involving the entries of the matrices involved). We illustrate this in terms of the NCAlgebra package, which is the main general noncommutative algebra package running under Mathematica. See Section VII. Computer algebra is of interest in its own right, and for some readers having this concrete manifestation of "abstract" noncommutative algebra might have tutorial benefit.

It is possible to use computer algebra to help automate checking for convexity, rather than depending upon lucky choices of X and Y as was done in the example above. The theory described in [7] and sketched later in Section IV leads to and validates a symbolic algorithm for determining regions of convexity of NC rational functions (NC rationals are formally introduced in Section III-A) which is currently implemented in NCAlgebra. We introduce the topic now with an example of an NCAlgebra command, leaving a more detailed discussion for later (see Section IV-A-I). The command is

$$\mathbf{NCConvexityRegion}\ [f, x]$$

where F is a noncommutative symmetric function of noncommutative variables $x = (x_1, \ldots, x_g)$.

Let us illustrate it on the example in the case x consists of (abusing notation slightly) the single variable x and the function is $p(x) = x^4$

$$\begin{aligned} \text{In[1]} &:= \text{SetNonCommutative[x];} \\ \text{In[2]} &:= \text{NCConvexityRegion}\ [x**x**x**x,\ x] \\ \text{Out[2]} &:= \text{"L** D** tp[L] gave non-trivial blocks} \\ &\qquad \text{so the output list is :} \\ &\{\{\text{diagonal}\}, \{\text{subdiagonal}\}, \{-\text{subdiagonal}\}\}\text{"} \\ &\qquad \{\{2,\ 0,\ 0\},\ \{0,\ 2\},\ \{0,\ -2\}\} \end{aligned}$$

which we interpret as saying that $p(x) = x^4$ is convex on the set of matrices X for which the the 3×3 block matrix-valued expression

$$\rho(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \tag{6}$$

is positive semidefinite. Thus, we conclude that p is *nowhere* convex. (In this simple example ρ is constant, not depending upon x.)

D. Road Map

Before continuing with the body of this survey, we pause to provide a guide to the remainder of the paper. Section II presents the formalism of noncommutative polynomials, linear matrix inequalities, and convexity. A heuristic principle says that noncommutative polynomials which satisfy very weak convexity hypotheses are in fact convex, have degree at most two, and can be described by a LMI. In Section III-A many of the results for polynomials are extended to noncommutative rational functions. Here domain issues play a major role. That the component of zero of the domain of a noncommutative symmetric rational function which is convex near the origin is itself convex and given by a LMI is a sample result. The starting point is a realization formula for noncommutative rational functions originally developed in the context of automata and formal languages. As an application it is possible to give a determinantal representation for *commutative* polynomials.

Semialgebraic geometry (SAG), both commutative and noncommutative, is at the core of the connection between positivity of a Hessian and convexity. SAG is discussed in Section IV where we also give a glimpse of the algorithm which goes into automatic convexity checking.

Sections V and VI deal with (commutative) variables in $\mathbb{R}^g$ and the questions of which convex sets in $\mathbb{R}^g$ either have a LMI representation or lift to a set with a LMI representation respectively. A reader who is already tired of matrix variables can read Sections V and VI now.

A list of available software can be found in Section VII. Section VIII places the whole survey in the larger context of the spectral theorem for bounded self adjoint linear operators. We collect broad conclusions, emerging principles, and grand conjectures in the conclusion, Section IX.

II. Dimension Free Case: Convexity

In this section we formalize the discussion of the Riccati inequality and noncommutative polynomials and present some sample theorems for convex symmetric polynomials in noncommuting variables, setting the stage for more general results considered later in this survey.

A. NC Polynomials

For the purposes of this article, we consider noncommutative polynomials, hereafter referred to simply as polynomials, in two classes of variables, $a = (a_1, \ldots, a_{g_a})$ and $x = (x_1, \ldots x_{g_x})$ which are taken to be formally symmetric as will be explained below in connection with the transpose operation ^{T}. The fact that the a variables are symmetric means that our class of polynomials does not include Riccatis, but the theory easily adapts to handle mixed symmetric and nonsymmetric variables which does include the Riccati. (The NCAlgebra software handles mixed variables.)

1) Example 2.1: The symmetrized Riccati (meaning the Riccati, but with a, b, c symmetric)

$$p(a,x) = a_1 x + x a_1 - x a_2^2 x + a_3^2$$

is, as expected, an example of such a polynomial. Here $a = (a_1, a_2, a_3)$ and $x = (x)$.

2) Example 2.2: For a second example, consider, for $a = (a_1, a_2)$ and $x = (x_1, x_2)$, the polynomial

$$q(a,x) = a_1 a_2 x_1 x_2 + a_1 x_2 + a_2 x_1. \tag{7}$$

There is a natural involution ^{T} on polynomials which acts on a monomial by reversing the order of the product which extends to polynomials by linearity. For instance, $(a_1 a_2 x_1 x_2)^T = x_2 x_1 a_2 a_1$ and, for the polynomial q from Example (2.2)

$$q^T(a,x) = x_2 x_1 a_2 a_1 + x_2 a_1 + x_1 a_2.$$

A polynomial p is *symmetric* if $p = p^T$. The polynomial in Example (2.1) is symmetric, while the polynomial from Example (2.2) is not. Note also, that $x_j^T = x_j$ for each j and so in this sense the variables themselves are symmetric.[2]

As the Riccati example suggests, given a polynomial $p(a,x)$, we wish to substitute matrices for the variables. Since the exposition is restricted to the case of formally symmetric variables, we substitute symmetric matrices. Accordingly, let $\mathbb{S}_n$ denote the set of symmetric $n \times n$ matrices, let $\mathbb{S}_n(\mathbb{R}^{g_a})$ denote the set of tuples $A = (A_1, \ldots, A_{g_a})$ with each $A_j \in \mathbb{S}_n$, and likewise for $X \in \mathbb{S}_n(\mathbb{R}^{g_x})$ and $(A,X) \in \mathbb{S}_n(\mathbb{R}^{g_a} \times \mathbb{R}^{g_x})$. Thus, for $(A,X) \in \mathbb{S}_n(\mathbb{R}^{g_a} \times \mathbb{R}^{g_x})$, substituting (A,X) for (a,x) produces the $n \times n$ matrix $p(A,X)$. Note that if p is symmetric, then $p(A,X)$ is also symmetric.

A symmetric polynomial $p(a,x)$ is **convex in** x provided that for each n, each $A \in \mathbb{S}_n(\mathbb{R}^{g_a})$ and $X, Y \in \mathbb{S}_n(\mathbb{R}^{g_x})$, and each $0 \le t \le 1$, the inequality

$$t\, p(A,X) + (1-t) p(A,Y) - p(A, tX + (1-t)Y) \succeq 0 \tag{8}$$

holds. (In particular, the left hand side is a symmetric matrix.) We emphasize that we allowing testing with matrices of all dimensions n.

A less stringent condition is convexity on a noncommutative convex set. For expository purposes, we will consider only the $\epsilon > 0$ x-neighborhood of 0 which is the sequence of sets, $\mathcal{N}(\epsilon) = (\mathcal{N}_n(\epsilon))_n$, given by

$$\mathcal{N}_n(\epsilon) = \{X \in \mathbb{S}_n(\mathbb{R}^{g_x}) : X_1^2 + \cdots + X_{g_x}^2 < \epsilon^2 I_n\}.$$

The polynomial p is convex in x on $\mathcal{N}(\epsilon)$ provided the inequality (10) holds for each n and each $A \in \mathbb{S}_n(\mathbb{R}^{g_a})$ and $X, Y \in \mathcal{N}_n(\epsilon)$. In this case, we say that p is **convex in** x **near 0**.

Theorem 2.3: If $p(a,x)$ is convex in x near 0, then p has degree at most two in x.

Theorem 2.4 says much more about the structure of p. The next subsection discusses linear pencils, a notion needed for its statement.

We close this section with several remarks. First, a notion of **convexity** for a class of noncommutative symmetric rational functions parallel to that for symmetric polynomials appears in Section III-A. Second, while the definitions of convex given here do not make such allowances, the theory readily adapts to include restrictions on a as well as on x. Third, in the definition of convex it is only necessary to check the definition for matrices up to size $n(d,g) \times n(d,g)$ for $n(d,g) = \sum_0^d g^j$, where d is the degree of the polynomial and $g = g_a + g_x$ is the number of variables. Finally, even in one variable, convexity differs from ordinary convexity, as evidenced by the example of $p(x) = x^4$ discussed in Section I-C.

B. NC Linear Pencils and LMIs

At the core of a LMI is a linear pencil. A $m \times m$ **NC linear pencil** (in g indeterminates) is an expression of the form

$$M(x) := M_0 + M_1 x_1 + \cdots + M_g x_g$$

where $M_0, M_1, \ldots, M_g$ are $m \times m$ symmetric matrices.[3] As an example, for

$$M_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_1 := \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad M_2 := \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix}$$

the pencil is

$$M(x) = \begin{pmatrix} 1 + 3x_1 + 5x_2 & 2x_1 + 4x_2 \\ 2x_1 + 4x_2 & -1 + x_1 + 2x_2 \end{pmatrix}.$$

[2]We could also work with not necessarily symmetric variables by introducing new variables y_j and declaring $x_j^T = y_j$ and $y_j^T = x_j$. For simplicity, the exposition is limited almost exclusively to the symmetric variable case and in the few cases not necessarily symmetric variables appear they are not important for the logical developments in the paper.

[3]The more general case of $m \times d$ (not square) linear pencils, or of square non symmetric pencils, is not needed here.

A (commutative) *linear matrix inequality (LMI)* is an inequality of the form $M(s) \succeq 0$, where M is a linear pencil, $s_j = x_j$ are real numbers (commutative), and $s = (s_1, \ldots, s_g)$.

The noncommutative setting necessitates substituting matrices, rather than scalars, for the variables. Accordingly, the entries of $M(x)$ are viewed as (noncommutative) polynomials in which case the corresponding inequality $M(X) \succeq 0$ will be referred to as a **noncommutative linear matrix inequality** or **NCLMI**. In fact, to incorporate the a variables, we consider the more general situation where

$$M = M(a,x) = (\, L_{j,l}(a,x)\,)_{j,l=1}^{N}$$

and each $L_{j,l}(a,x)$ is a polynomial of degree at most one in x. This linear pencil in x is symmetric if $M^T = M$ in the sense that $L_{j,l}^T = L_{l,j}$. We will refer to the inequality $M(A,X) \succeq 0$ as a **NCLxMI**. Of course, what is of interest is the feasible set of the inequality. Namely, $\cup_n \{(A,X) \in \mathbb{S}_n(\mathbb{R}^{g_a} \times \mathbb{R}^{g_x}) : M(A,X) \succeq 0\}$.

As a canonical example of an NCLxMI, given polynomials $\Lambda_j(a,x)$ for $j = 1, \ldots, m$, which are linear in x and a polynomial $L(a,x)$ which has degree at most one in x, let

$$M(a,x) = \begin{pmatrix} I_m & \Lambda(a,x) \\ \Lambda(a,x)^T & L(a,x) \end{pmatrix} \tag{9}$$

where I_m is the $m \times m$ identity matrix and $\Lambda(a,x)$ is the column vector with entries $\Lambda_j(a,x)$. By taking Schur complements as in the Riccati example, the inequality $M(A,X) \succeq 0$ is equivalent to the inequality

$$\begin{aligned} 0 \preceq q(A,X) &:= L(A,X) - \Lambda(A,X)^T \Lambda(A,X) \\ &= L(A,X) - \sum \Lambda_j(A,X)^T \Lambda_j(A,X). \end{aligned} \tag{10}$$

By comparison, an example of a Noncommutative matrix inequality (NCMI) is an expression of the form $M(A,X) \succeq 0$, where M is a square matrix whose entries $M_{j,\ell}$ are NC polynomials such that $M_{j,\ell}^T = M_{\ell,j}$.

C. NC Convex Polynomials are Trivial

The polynomial q of the previous subsection is the Schur complement of a NCLxMI, and is therefore concave in x (meaning $-q(a,x)$ is convex in x). The converse of this later statement is the main result on convex polynomials.

Theorem 2.4: If the symmetric polynomial $p(a,x)$ is convex in x near 0, then p has degree at most two in x. Moreover:

1) p is convex in x (everywhere);
2) $-p$ is the Schur complement of a NCLxMI $M(a,x)$ of the form in (9) so that $p(a,x) = -L(a,x) + \sum_{j=1}^{n} \Lambda_j(a,x)^T \Lambda_j(a,x)$ where $L(a,x)$ is symmetric and has degree at most one in x, and the polynomials $\Lambda_j(a,x)$ are linear in x;
3) the feasible set of the inequality $-p(A,X) \succeq 0$ is convex in x.

There are a number of refinements of Theorem 2.4 of which the following is a sample. For psychological reasons we now use the variables (x,y).

Theorem 2.5: If $p(x,y)$ is convex in x and concave in y, then there exist:

1) integers μ, ν and polynomials $q_1(x), \ldots, q_\mu(x)$ and $r_1(y), \ldots, r_\nu(y)$ which are linear in x and y, respectively;
2) a polynomial $L(x,y)$ which has degree at most one in each of x and y so that

$$p(x,y) = L(x,y) + \sum q_j(x)^T q_j(x) - \sum r_j(y)^T r_j(y).$$

Remark 2.6: Both theorems can be found in [15]. The proofs rely on a careful analysis of the Hessian— a type of noncommutative second derivative. Very weak hypothesis about positivity of the Hessian are enough to reach the conclusions of the Theorem. Further details can be found in Section IV and the references therein.

A discussion of the matrix version of Theorem 2.5 and other variations on topics in this section can be found in Section 10 of [21].

For the case that there are no a variables Theorem 2.4 goes back to [16]. The further special case of one variable seems to be a folk theorem.

D. Noncommutative Convex Sets

Before taking up noncommutative rational functions in the next section we wish to connect the discussion of convex polynomials to the relation between convex NCMIs and NCLMIs.

A special case of a matrix valued NCMI is the intersection of a finite collection of NCMIs. Given a finite set of symmetric polynomials, $\mathcal{P} = \{p_1(a,x), \ldots, p_k(a,x)\}$, let

$$\mathcal{P}_{A,n}^+ = \{X \in \mathbb{S}_n(\mathbb{R}^{g_a} \times \mathbb{R}^{g_x}) : p_j(A,X) \succeq 0,\ j = 1, 2, \ldots, k\} \tag{11}$$

and consider the joint feasible (positivity) set $\mathcal{P}^+ = (\mathcal{P}_{A,n}^+)$.

The feasible set of a family NCLxMIs is evidently convex. We conjecture that the converse is true for $\mathcal{P}_{A,n}^+$.

1) Conjecture 2.7: If $\mathcal{P}^+$ is convex, then there is a NCLxMI $L = L(a,x)$ such that $\mathcal{P}_{A,n}^+$ is the feasible set of $L(A,X) \succeq 0$. The guiding heuristic is that convex NCMIs are no more general than NCLMIs. The conjecture also makes sense for nonsymmetric a variables or with a convex side constraint on a.

Evidence for the conjecture is:

1) If we do not have a variables as in 11 and we assume that $p_j(0) > 0$ for each j, then a theorem of Effros and Winkler [11] applies with the conclusion that there is a collection $\mathcal{L}$, possibly infinite, of NCLMIs M which are monic ($M(0) = I$) and such that $\mathcal{P}_n^+ = \mathcal{L}_n^+$. In other words, $\mathcal{P}_n^+$ is the feasibility set for a (possibly infinite) family of monic NCLMIs.
2) Theorem 2.4, and its generalization to symmetric rational functions taken up in the next section, says, under stronger assumptions, that the collection $\mathcal{L}$ promised by Effros and Winkler can be chosen to be a singleton; i.e., $(\mathcal{P}_n^+)_n$ is given by a single NCLMI.
3) The main result in [10] says, if $\mathcal{P} = \{p\}$ is a singleton and under irreducibility and regularity conditions on p and with hypothesis slightly stronger than convex on $\mathcal{P}^+$, then p is concave.

III. Convexity for NC Rationals

In this section we provide more evidence for Conjecture 2.7 by describing the extension of the convex NC polynomial theorem, Theorem 2.4, to symmetric NC rational functions, $r = r(x)$, of the x variables alone which are convex near the origin, see [18].

A. NC Rational Functions

We shall discuss the notion of a NC rational function in terms of rational expressions, referring to [18, Sections 2 and 16] for details. (We consider only rational expressions and functions that are analytic at 0, and we will not mention it explicitly in the sequel.)

A **NC rational expression** is defined recursively. NC polynomials are NC rational expressions as are all sums and products of NC rational expressions. If r is a NC rational expression and $r(0) \neq 0$, then the inverse of r is a rational expression.

The notion of the**formal domain of a rational expression**r, denoted $\mathcal{F}_{r,\text{formal}}$, and the evaluation $r(X)$ of the rational expression at a tuple $X \in \mathcal{F}_{r,\text{formal}}$ are also defined recursively.[4] Example (3.1) below is illustrative.

An example of a NC rational expression is the Riccati expression for discrete-time systems

$$r = a^T xa - x + c^T c \\ +(a^T xb + c^T d)(I - d^T d - b^T xb)^{-1}(b^T xa + d^T c).$$

Here some variables are symmetric and some are not. A difficulty is that it is possible for two different expressions, such as $r_1 = x_1(1 - x_2 x_1)^{-1}$ and $r_2 = (1 - x_1 x_2)^{-1} x_1$, to be converted into each other with algebraic manipulation and thus represent the same function and one needs to specify an equivalence relation on rational expressions to arrive at what are typically called **NC rational functions**. (This is standard and simple for commutative (ordinary) rational functions.) There are many alternate ways to describe the NC rational functions and they go back 50 years or so in the algebra literature. The simplest one for our purposes is **evaluation equivalence**— two rational expressions r_1 and r_2 are evaluation equivalent if $r_1(X) = r_2(X)$ for all $X \in \mathcal{F}_{r_1,\text{formal}} \cap \mathcal{F}_{r_2,\text{formal}}$. For engineering purposes one need not be too concerned, since what happens is that two expressions r_1 and r_2 are equivalent whenever usual matrix manipulations convert r_1 to r_2.

Define the **domain** of a rational function (equivalence class of rational expressions r) by

$$\mathcal{F}_{\mathsf{r}} := \cup_{\{r \text{ } represents \text{ } \mathsf{r}\}} \mathcal{F}_{r,\text{formal}}.$$

Let $\mathcal{F}_{\mathsf{r}}^0$ denote the arcwise connected component of $\mathcal{F}_{\mathsf{r}}$ containing 0 (and similarly for $\mathcal{F}_{r,\text{formal}}^0$). We call $\mathcal{F}_{\mathsf{r}}^0$ the **principal component** of $\mathcal{F}_{\mathsf{r}}$. Henceforth we do not distinguish between the rational functions r and rational expressions r, since this causes no confusion. Several examples follow.

[4] The formal domain of a polynomial p is all of $\mathbb{S}_n(\mathbb{R}^{g_x})$ and $p(X)$ is defined just as before. The formal domain of sums and products of rational expressions is the intersection of their respective formal domains. If r is an invertible rational expression and $r(X)$ is invertible, then X is in the the formal domain of r^{-1}.

1) Example 3.1: $r(x_1, x_2) = (1 + x_1 - (3 + x_2)^{-1})^{-1}$ is a symmetric NC rational expression. The domain $\mathcal{F}_{r,\text{formal}}$ is

$$\cup_{n>0}\{(X_1, X_2) \in \mathbb{S}_n(\mathbb{R}^2) : \\ 1 + X_1 - (3 + X_2)^{-1} \text{ and } 3 + X_2 \text{ are invertible}\}.$$

Its principal component $\mathcal{F}_r^0$ is

$$\cup_{n>0}\{(X_1, X_2) \in \mathbb{S}_n(\mathbb{R}^2) : 1 + X_1 - (3 + X_2)^{-1} > 0 \\ \text{and } 3 + X_2 > 0\}$$

2) Example 3.2: We return to the convexity checker command and illustrate it on

$$F((a, b, r), (x, y)) := -(y + a^T xb)(r + b^T xb)^{-1}(y + b^T xa) \\ +a^T xa. \quad (12)$$

Here we are viewing F as a function of the two classes of variables (x, y) and (a, b) with the first symmetric $(x = x^T, y = y^T)$ and the second not. An application of the command **NCConvexityRegion** $[F, \{x, y\}]$ outputs the list $\{-2\,(r + b^T xb)^{-1},\ 0,\ 0,\ 0\}$.

This output has the meaning that whenever A, B, R are fixed matrices, the function F is "x, y concave" on the domain

$$\mathcal{G}_{A,B,R} := \{(X, Y) : (R + B^T XB)^{-1} \succ 0\}.$$

The command NCConvexityRegion also has an important feature which, for this problem, assures us no domain bigger than

$$\bar{\mathcal{G}}_{A,B,R} := \{(X, Y) : R + B^T XB \succeq 0\}$$

is a "domain of concavity" for F. The algorithm is discussed briefly in Section IV. For details and proof of the last assertion, see [7].

B. Convexity vs LMIs

In the case that there are no a variables, the following theorem characterizes symmetric NC rational functions (in x) which are convex near the origin in terms of a LMI.

Theorem 3.3: [18]If $r = r(x)$ is a NC symmetric rational function of g variables which is convex (in x) near the origin, then

1) there is a linear pencil $L(x)$ such that the set $\{X : I - L(X) \succ 0\}$ is the principal component of the domain of r. In particular, the principal component of domain r is convex;
2) r is convex on the principal component of its domain;
3) there exists n and m, symmetric $n \times n$ matrices $A_1, \ldots, A_g$, a $1 \times n$ vector $\Lambda(x)$ of linear noncommutative polynomials (so a column vector with entries $\Lambda_j(x)$), a $1 \times m$ vector $\ell(x)$ of linear noncommutative polynomials, and a symmetric polynomial $r_1(x)$ which has degree at most one in x so that r has the representation

$$r(x) = r_1(x) + \ell(x)\ell(x)^T + \Lambda(x)(I - L(x))^{-1}\Lambda(x)^T \quad (13)$$

where $L(x) = \sum_{j=1}^{g} A_j x_j$. Moreover, L can be chosen as in (1).

Thus, for a given real number γ, $r-\gamma$ is a Schur complement of the noncommutative linear pencil

$$\mathcal{L}_\gamma(x) := \begin{pmatrix} -1 & 0 & \ell(x)^T \\ 0 & -(I-L(x)) & \Lambda(x)^T \\ \ell(x) & \Lambda(x) & r_1(x)-\gamma \end{pmatrix}.$$

Conversely, if r has the form (13), then r is convex.

This correspondence between properties of the pencil and properties of r yields

Corollary 3.4: For $\gamma \in \mathbb{R}$, the principal component, $\mathcal{G}_\gamma^0$, of the set of solutions X to the NCMI $r(X) \prec \gamma I$ equals the set of solutions to a NCLMI based on a certain linear pencil $\mathcal{L}_\gamma(x)$.

That is, **numerically solving matrix inequalities based on** r **is equivalent to numerically solving a NCLMI associated to** r.

Proof of Corollary 3.4

By item (2) of Theorem 4.2 the upper 2×2 block of $\mathcal{L}_\gamma(X)$ is negative definite if and only if $I - L(X) \succ 0$ if and only if X is in the component of 0 of the domain of r. Given that the upper 2×2 block of $\mathcal{L}_\gamma(X)$ is negative definite, by the LDL^T (Cholesky) factorization, $0 \succ \mathcal{L}_\gamma(X)$ is negative definite if and only if $\gamma I \succ r(X)$. □

Proof of Theorem 3.3

The proof consists of several stages and is in principle constructive. It is interesting that the technique for the first stage, which yields an initial representation for r as a Schur Complement of a linear pencil, is classical. In fact, the following representation of any symmetric NC rational function r is the symmetric version of the one due originally to Kleene, Schützenberger, and Fliess (who were motivated by automata and formal languages, and bilinear systems; see [4] for a good survey), and further studied recently by Beck [2], see also [3], [29], and by Ball–Groenewald–Malakorn, e.g. [1].

A rational function r has a symmetric **Descriptor, or Recognizable Series, Realization** if there is an n, $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_{g_x}) \in \mathbb{S}_n(\mathbb{R}^{g_x})$, a matrix J such that $J = J^T$ and $J^2 = I$, a vector $C \in \mathbb{R}^n$, and a scalar r_0 such that

$$r(x) = r_0 + C\left(J - \sum_{j=1}^{g_x} \mathcal{A}_j x_j\right)^{-1} C^T. \qquad (14)$$

The realization is **minimal** if $CJ\mathcal{A}_{i_1}\cdots J\mathcal{A}_{i_k} v = 0$ for all k and choices of $i_j \in \{1, \ldots, g_x\}$ implies $v = 0$.

Theorem 3.5: If r is a NC symmetric rational function (which is analytic at 0), then r has a minimal symmetric descriptor realization.

Of course in general the above symmetric realization is not monic, i.e., $J \neq I$. The second stage of the proof uses the convexity of r near the origin, more precisely, the positivity of the NC Hessian, to force J to be within rank one of I. Algebraic manipulations then give Theorem 3.3 item (1).

The third stage of the proof—establishing item (2)—was quite grueling in [18], but it is subsumed now under the following fairly general singularities theorem for various species of minimal NC realizations, see [24].

Theorem 3.6: Suppose

$$r(x) = d(x) + C(x)\left(I - \sum_{j=1}^{g} A_j x_j\right)^{-1} B(x) \qquad (15)$$

where $d(x)$ is a NC polynomial, $A_j \in \mathbb{R}^{n\times n}$, and $B(x) = \sum B_{j_1\ldots j_r} x_{j_1}\cdots x_{j_r}$ and $C(x) = \sum C_{j_1\ldots j_l} x_{j_1}\cdots x_{j_l}$ are $n\times 1$ and $1\times n$ matrix valued NC polynomials, homogeneous of degrees r and l, respectively. Assume the "minimality type" conditions

$$\underset{k\geq 0;\ 1\leq i_1,\ldots,i_k,\ j_1,\ldots,j_r \leq g_x}{\operatorname{span}} \operatorname{ran} A_{i_1}\cdots A_{i_k} B_{j_1\ldots j_r} = \mathbb{R}^n$$

$$\bigcap_{k\geq 0;\ 1\leq i_1,\ldots,i_k\ j_1,\ldots,j_l\leq g_x} \ker C_{j_1\ldots j_l} A_{i_1}\cdots A_{i_k} = 0.$$

Then

$$\mathcal{F}_r = \{(X_1,\ldots,X_{g_x}) : \det(I - A_1\otimes X_1 - \cdots A_{g_x}\otimes X_{g_x}) \neq 0\}.$$

The proof is based on the formalism of NC backward shifts and thus the theorem applies more generally to matrix-valued NC rational functions.

IV. NC Semialgebraic Geometry (SAG)

A NC symmetric polynomial p is a **matrix positive polynomial** or simply **positive** provided $p(X_1, \ldots, X_g)$ is positive semidefinite for every $X \in \mathbb{S}_n(\mathbb{R}^{g_x})$ (and every n). An example of a positive NC polynomial is a **Sum of Squares** of NC polynomials, meaning an expression of the form

$$p(x) = \sum_{j=1}^{c} h_j(x)^T h_j(x).$$

Substituting $X \in \mathbb{S}_n(\mathbb{R}^{g_x})$ gives $p(X) = \sum_{j=1}^{c} h_j(X)^T h_j(X) \succeq 0$. Thus p is positive. Remarkably these are the only positive NC polynomials.

Theorem 4.1: Every positive NC polynomial is a sum of squares.

Those familiar with conventional "commutative" semialgebraic geometry will recognize that this NC behavior is much cleaner. See [25], [33] for a beautiful treatment of applications of commutative SAG. For nonsymmetric variables, the result is due to Helton [14]. The version here for symmetric (matrix-valued polynomials) is from [30].

This theorem is just a sample of the structure of NC semialgebraic geometry. Indeed for those inclined toward SAG we mention that there is a strong NC Positivstellensätz. References containing this and the results alluded to in this section can be found in the bibliography of [17] and the survey [21].

A. Application of NC SAG to NC Convexity

It is easy to show that convexity of an NC rational function on a "convex domain" is equivalent to its NC second directional derivative being positive. This is the link between NC convexity and NC positivity (i.e., NCSAG). For illustration we show $p(x) = x^4$ is not convex.

1) Symbolic Differentiation of NC Functions: The *first directional derivative* of a noncommutative rational function $r(a, x)$ with respect to x in the direction h is defined in the usual way

$$\frac{\partial}{\partial x} r(a,x)[h] := \frac{d}{dt}\, r(a, x+th)\big|_{t=0}.$$

Likewise, the *second directional derivative* is

$$\frac{\partial^2}{\partial^2 x} r(a,x)[h] = \frac{d^2}{dt^2} r(a, x+th)\Big|_{t=0}. \tag{16}$$

When there are no a variables, we write, as one would expect, $r'(x)[h]$ and $r''(x)[h]$ instead of $\frac{\partial}{\partial x} r(x)$ and $\frac{\partial^2}{\partial^2 x} r(x)$.

As an example, choose $x = (x = x_1)$ and $p(x) = x^4$. The directional derivatives of p are then

$$p'(x)[h] = hxxx + xhxx + xxhx + xxxh$$
$$p''(x)[h] = 2(hhxx + hxhx + hxxh + xhhx + xhxh + xxhh).$$

2) Nonconvexity of x^4: Arguing by contradiction, if $p(x) = x^4$ is convex , then $p''(x)[h]$ is positive and therefore, by Theorem 4.1, there exists a k and polynomials $f_1(x,h), \ldots, f_k(x,h)$ so that

$$hhxx + hxhx + hxxh + xhhx + xhxh + xxhh = f_1(x,h)^T f_1(x,h) + \cdots + f_k(x,h)^T f_k(x,h).$$

One can show that each $f_j(x,h)$ is linear in h. On the other hand, some term $f_i^T f_i$ contains $hhxx$ and thus f_i contains hx^2. Let m denote the largest ℓ such that some f_j contains the term hx^ℓ. Then $m \geq 1$ and for such j, the product $f_j^T f_j$ contains the term $hx^{2m}h$ which can't be cancelled out, a contradiction. □

3) Middle Matrix: There is a canonical representation of rational functions $q(x)[h]$ which are homogeneous of degree two in h as a matrix product

$$q(x)[h] = V(x)[h]^T Z(x) V(x)[h]. \tag{17}$$

In the case that $r(a,x)$ is a polynomial of degree d, and $q(a,x)[h] := \frac{\partial^2}{\partial^2 x} r(a,x)[h]$, then $V(a,x)[h]$ is a (column) vector whose entries are monomials of the form $h_j m(a,x)$ where $m(a,x)$ is a monomial in the variables (a,x) of degree at most $d-1$ (each such monomial appearing exactly once), and where $Z(a,x)$ is a matrix whose entries are polynomials in (a,x). The matrix Z is unique, up to the order determined by the ordering of the $h_j m(a,x)$ in $V(a,x)[h]$. The matrix $Z(a,x)$ is the called the **middle matrix** and $V(a,x)[h]$ the **border** or **tautological vector**.

For $p(x) = x^4$, the decomposition of (17) is given by

$$p''(x)[h] = 2\begin{pmatrix} h & xh & x^2h \end{pmatrix} \begin{pmatrix} x^2 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ hx \\ hx^2 \end{pmatrix}. \tag{18}$$

It turns out that a variety of very weak hypotheses on positivity of $\frac{\partial^2}{\partial^2 x} r(a,x)[h]$ imply positivity of the middle matrix. For instance, for a polynomial $p(a,x)$, we have $\frac{\partial^2}{\partial^2 x} p(A,X)[H] \succeq 0$ for X near 0 and all A and H if and only if $Z(A,X) \succeq 0$ for X near 0. In particular, since it is evident that the middle matrix for the polynomial $p(x) = x^4$ is not positive semidefinite (for any X) its Hessian is not positive semidefinite near 0 and hence p is not convex.

4) Automated Convexity Checking: The example of $p(x) = x^4$ illustrates

Our Convexity Algorithm for an NC rational r

1) Compute symbolically the Hessian $q(a,x)[h] := \frac{\partial^2}{\partial^2 x} r(a,x)[h]$.
2) Represent $q(a,x)[h]$ as $q(a,x)[h] = V(a,x)[h]^T Z(a,x) V(a,x)[h]$.
3) Apply the noncommutative LDL ^{T} decomposition to the matrix $Z(a,x)$, to get $Z(a,x) = LDL^T$. When r is convex (anywhere) the matrix $D(a,x)$ is diagonal, so has the form $D = diag\,\{\rho_1(a,x), \ldots, \rho_c(a,x)\}$.
4) The Hessian $q(A,X)[H]$ is positive semidefinite if $D(A,X) \succeq 0$ (for all H). Thus a set $\mathcal{D}$ where r is convex is given by

$$\mathcal{D} = \{(A,X) : \rho_j(A,X) \succeq 0, \quad j = 1, \ldots, c\}. \tag{19}$$

(In the case of (18), $D(X)$ equals (6) which is not diagonal. In particular, convexity fails.)

The surprising and deep feature is that (under very weak hypotheses) the closure of $\mathcal{D}$ is the largest possible domain of convexity. See [7] for the proof, though facets of the proof have been extended with streamlined proofs, cf. [18].

It is hard to imagine a precise "convexity region algorithm" not based on noncommutative calculations, the problem being that matrices of practical size often have thousands of entries, so would lead to calculations with huge numbers of polynomials in thousands of variables.

V. Dimension Dependent Case: Convexity

In this section and the next one we treat commutative variables, which to avoid confusing with our noncommutative variables, we denote by $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_g)$ and corresponding linear pencils $M(\mathtt{x})$.

A. Two Closely Related Questions

Q1. A set $\mathcal{C} \subset \mathbb{R}^g$ has a **LMI Representation** provided there is a linear pencil $M(\mathtt{x})$ for which

$$\mathcal{C} = \{\mathtt{x} \in \mathbb{R}^g : M(\mathtt{x}) \quad \text{is PosSemiDef}\}. \tag{20}$$

We assume that the set $\mathcal{C}$ has a nonempty interior, and if we make a normalization of 0 being an interior point of $\mathcal{C}$, then without loss of generality ([22, Lemma 2.3]) we may take the pencil to be monic, that is $M(0) = I$. Parrilo and Sturmfels (see [35]) formally ask: **Which sets have a LMI representation?**

Q2. A polynomial $\check{p}$ has a **monic determinantal representation** if there is a monic pencil M so that

$$\check{p}(\mathtt{x}) := \kappa \det[I + M_1\mathtt{x}_1 + \cdots + M_g\mathtt{x}_g]. \tag{21}$$

where the M_j are symmetric $d \times d$ matrices, and κ is a constant. This representation is said to be a **monic determinantal representation** for $\check{p}$ and d is called the **size** of the representation. **Which polynomials have a monic determinantal representation?** A version of this question for homogeneous polynomials was posed already by Lax [27] in 1958.

As we shall see an answer to Q2 also answers Q1. This section describes work in [22] which settles the questions for $g = 2$.

B. Obvious Necessary Conditions

1) Real Zero Polynomials: For a determinantal representation to exist there is an obvious necessary condition. Observe from (25) that

$$\check{p}(\mu \mathrm{x}) := \mu^d \det \left[\frac{I}{\mu} + M_1 \mathrm{x}_1 + \cdots + M_g \mathrm{x}_g \right].$$

Since (for real numbers x_j) all eigenvalues of the symmetric matrix $M_1\mathrm{x}_1 + \cdots + M_g\mathrm{x}_g$ are real we see that, while $p(\mu\mathrm{x})$ is a complex valued function of the complex variable μ, it vanishes only at μ which are real numbers. This condition is critical enough that we formalize it in a definition.

A (real) polynomial p satisfies the **real zeros condition (RZ)** if for each $\mathrm{x} \in \mathbb{R}^g$ the polynomial $p_{\mathrm{x}}(\mu) = p(\mu\mathrm{x})$ of the complex variable μ has only real zeros. A **Real Zero polynomial** (**RZ** polynomial) is a polynomial satisfying the real zeros condition.

Example 5.1: For $p(\mathrm{x}) := 1 - (\mathrm{x}_1^4 + \mathrm{x}_2^4)$ and $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^2$

$$p(\mu\mathrm{x}) = [1 - \mu^2(\mathrm{x}_1^4 + \mathrm{x}_2^4)^{1/2}][1 + \mu^2(\mathrm{x}_1^4 + \mathrm{x}_2^4)^{1/2}]$$

has two complex (not real) zeroes $\mu_\pm := i(\mathrm{x}_1^4 + \mathrm{x}_2^4)^{-1/2}$. Thus p does not satisfy the Real Zero condition.

There is of course a similar Real Zero condition $\mathbf{RZ}_{\mathrm{x}^0}$ with respect to any given point $\mathrm{x}^0 \in \mathbb{R}^g$, where we consider the zeroes of $p(\mathrm{x}^0 + \mu\mathrm{x})$ for each $\mathrm{x} \in \mathbb{R}^g$. As noticed by Lewis, Parrilo and Ramana [28], **RZ** polynomials are simply the nonhomogeneous version of the hyperbolic polynomials that were first introduced by Petrowski and Garding in the study of PDEs.

2) LMI Representations and Algebraic Interiors: Suppose we are given a monic pencil $M(\mathrm{x})$ which represents a set $\mathcal{C}$ as in (20). Clearly:

i) $\mathcal{C}$ is a convex set (with 0 an interior point of $\mathcal{C}$).
ii) The polynomial $\check{p}(\mathrm{x}) := det(M(\mathrm{x}))$ is positive on the interior of $\mathcal{C}$ and vanishes on the boundary of $\mathcal{C}$.

A closed subset C of $\mathbb{R}^g$ is called an **Algebraic Interior** if it equals the closure of a connected component of $\{\mathrm{x} \in \mathbb{R}^g : p(\mathrm{x}) > 0\}$ for some polynomial p in g variables. In this case p is called a **defining polynomial** for C. Therefore the set $\mathcal{C}$ is an algebraic interior with a defining polynomial $\check{p}$. It is shown in [22] that:

The minimal degree defining polynomial p for an Algebraic Interior C is unique (up to a positive constant factor), and divides any other defining polynomial for C. In fact, p is the defining polynomial of the Zariski closure of the boundary of C in $\mathbb{R}^g$. If d denotes the degree of p, we say that C is an **Algebraic Interior of Degree** d.

3) Geometrical Version of the Necessary Conditions: Rigid Convexity: Consider an algebraic interior C of degree d in $\mathbb{R}^g$ with minimal defining polynomial p. Then C will be called **rigidly convex** provided that, for every point x^0 in the interior of C and for a generic line ℓ through 0, ℓ intersects the (affine) real algebraic hypersurface $p(\mathrm{x}) = 0$ in exactly d points.[5]

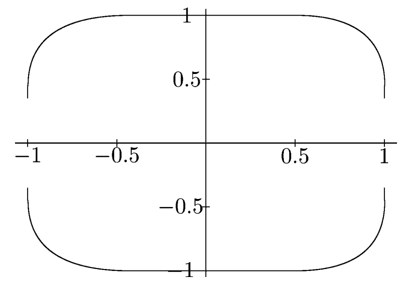

Fig. 1. $p(\mathrm{x}_1, \mathrm{x}_2) = 1 - \mathrm{x}_1^4 - \mathrm{x}_2^4$.

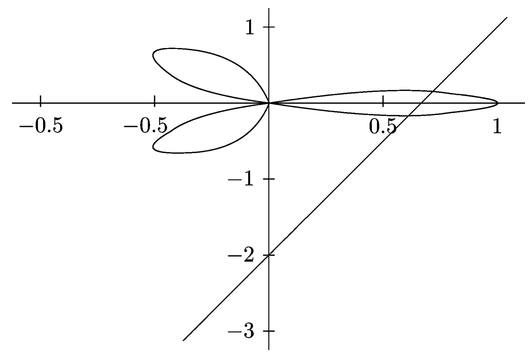

Fig. 2. A line thru $O = (0, 0.7)$ hitting $\mathrm{x}_1^3 - 3\mathrm{x}_2^2\mathrm{x}_1 - (\mathrm{x}_1^2 + \mathrm{x}_2^2)^2 = 0$ in only 2 points.

Proposition 5.2 (see [22]):

1) If the line test which defines rigid convexity holds for one point x^0 in the interior of C, then it holds for all points in the interior of C, thereby implying rigid convexity.
2) A rigidly convex algebraic interior is convex.
3) Rigid convexity of an Algebraic Interior C containing 0 as an interior point is the same as its minimal degree defining polynomial p having the Real Zero Property.

Example 5.3 (Example 5.1 Revisited): $p(\mathrm{x}_1, \mathrm{x}_2) = 1 - \mathrm{x}_1^4 - \mathrm{x}_2^4$. The convex algebraic interior $C := \{\mathrm{x} : p(\mathrm{x}) \geq 0\}$ has degree 4 (since p is irreducible it is the minimum defining polynomial for C), but all lines in $\mathbb{R}^2$ through an interior point intersect the set $p = 0$ in exactly two places. Thus C is not rigidly convex (see Fig. 1).

Example 5.4: Fig. 2 shows the zero set of $p(\mathrm{x}_1, \mathrm{x}_2) = \mathrm{x}_1^3 - 3\mathrm{x}_2^2\mathrm{x}_1 - (\mathrm{x}_1^2 + \mathrm{x}_2^2)^2$.

The complement to $p = 0$ in $\mathbb{R}^2$ consists of four components, three bounded convex components where $p > 0$ and an unbounded component where $p < 0$. Let us analyze one of the bounded components, say the one in the right half plane, C is the closure of

$$\{\mathrm{x} : p(\mathrm{x}) > 0, \mathrm{x}_1 > 0\}.$$

Is C rigidly convex? To check this: fix a point O inside C, e.g., $O = (.7, 0)$.

[5]One can replace here "almost every line" by "every line" if one takes multiplicities into account when counting the number of intersections, and also counts the intersections at infinity, i.e., replaces the affine real algebraic hypersurface $p(\mathrm{x}) = 0$ in $\mathbb{R}^g$ by its projective closure in $\mathbb{P}^g(\mathbb{R})$.

We can see from the picture in $\mathbb{R}^2$ that there is a continuum of real lines ℓ through $(0, 0.7)$ intersecting $p = 0$ in exactly two real points. Thus C is not rigidly convex.

C. Main Representation Results

Theorem 5.5: If $\mathcal{C}$ has a LMI representation, then it is rigidly convex. The converse is true in two dimensions, that is, when $g = 2$. Furthermore, in this case there exists a LMI representation of the size equal to the degree of $\mathcal{C}$.

The direct implication is clear in view of the previous subsection: if $\mathcal{C}$ is represented by a monic pencil $M(\mathtt{x})$ then $\mathcal{C}$ is an algebraic interior with defining polynomial $\check{p}(\mathtt{x}) = \det M(\mathtt{x})$ which satisfies the Real Zero condition; therefore the minimal degree defining polynomial p of $\mathcal{C}$, which is a factor of $\check{p}$, also satisfies the Real Zero condition, and $\mathcal{C}$ is rigidly convex.

The converse when $g = 2$ is established by showing that that any Real Zero polynomial of degree d in two variables admits a monic determinantal representation of size d. The proof uses classical one-dimensional algebraic geometry on the complexification of the projective closure of the the (affine) real algebraic curve $p(\mathtt{x}) = 0$. There are two possible constructions, one using theta functions and the Abel-Jacobi map, and one using bases of certain linear systems, more concretely, curves of degree $d-1$ that are everywhere tangent to the original curve. The actual computational aspects of both constructions remain to be explored.

Conjecture 5.6: For any dimension g, if $\mathcal{C}$ is rigidly convex, then $\mathcal{C}$ has a LMI representation.

Remark 5.7: When $g > 2$, the minimal size of a LMI representation in Conjecture 5.6 is in general larger than the degree of $\mathcal{C}$.

Conjecture 5.6 follows from

Conjecture 5.8 ([22]): Every Real Zero polynomial has a monic determinantal representation.

This might be called a modified Lax conjecture in that Lewis, Parrilo and Ramana [28] settled a 1958 conjecture of Peter Lax affirmatively for $g = 2$, using the $g = 2$ monic determinantal representation mentioned above. They noticed that the Lax conjecture is false for $g > 2$, but our conjecture is a natural modification. Another modification of the Lax conjecture for $g > 2$, using mixed determinants, has been proposed recently by Borcea, Branden and Shapiro [6].

A bit of evidence for Conjecture 5.8 is provided by

Theorem 5.9 ([18]): Every polynomial $\check{p}$ (with $\check{p}(0) \neq 0$) admits a symmetric determinantal representation,

$$\check{p}(\mathtt{x}) = \kappa \det[J + M_1\mathtt{x}_1 + \cdots + M_g\mathtt{x}_g] \qquad (22)$$

with J a "signature matrix", that is, J is symmetric and $J^2 = I$, M_j are symmetric matrices, and κ is a constant.

Of course in general the symmetric determinantal representation in Theorem 5.9 is not monic, i.e., $J \neq I$.

The proof of Theorem 5.9 in [18] uses *noncommutative techniques* like those in Section III-A.

More precisely, what is actually proved is the following theorem.

Theorem 5.10: Every symmetric NC polynomial p (with $p(0) \neq 0$) admits a NC symmetric determinantal representation, i.e., there exist a signature matrix J, a constant κ, and symmetric matrices M_j such that

$$\det p(X_1, \ldots, X_g) = \kappa \det[J \otimes I + M_1 \otimes X_1 + \cdots + M_g X_g] \qquad (23)$$

for all tuples of symmetric (in fact, of arbitrary square) matrices $(X_1, \ldots, X_g)$.

This implies immediately Theorem 5.9 by taking p to be an arbitrary NC lifting of the given commutative polynomial $\check{p}$, and substituting scalar (rather than matrix) variables.

The proof of Theorem 5.10 is based on the symmetric realization theorem (Theorem 3.5): roughly, a realization of p^{-1} yields a symmetric determinantal representation of p. Notice that this fits with the singularities theorem (Theorem 3.6), since the domain of p^{-1} is $\{X : \det p(X) \neq 0\}$.

VI. LMI Lifts

As we saw in the previous section not every convex semialgebraic set S has a LMI representation. A $S \subset \mathbb{R}^g$ lifts to a set $\hat{S} \subset \mathbb{R}^{g+N}$ if S is the image of $\hat{S}$ under the coordinate projection of $\mathbb{R}^{g+N}$ onto $\mathbb{R}^g$. The set $S \subset \mathbb{R}^g$, is *semidefinite representable* or *SDP representable* if there exist an N and a LMI representable set $\hat{S} \subset \mathbb{R}^{g+N}$ such that S lifts to $\hat{S}$. The original book by Nesterov and Nemirovski ([31]) gave collections of examples of SDP representable sets, thereby leading to the question: which sets are SDP representable? In Section 4.3.1 of his excellent 2006 survey [32] Nemirovsky said " this question seems to be completely open."

Let us look at an example. Recall that the convex set in Example 5.1

$$T := \{\mathtt{x} \in \mathbb{R}^2 : 1 - (\mathtt{x}_1^4 + \mathtt{x}_2^4) \geq 0\}$$

does not admit a LMI representation [22], since it is not rigidly convex. However, the set T is the projection onto $\mathtt{x}$-space of the set

$$\hat{S} := \left\{(\mathtt{x}, \mathtt{w}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \begin{bmatrix} 1 + \mathtt{w}_1 & \mathtt{w}_2 \\ \mathtt{w}_2 & 1 - \mathtt{w}_1 \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & \mathtt{x}_1 \\ \mathtt{x}_1 & \mathtt{w}_1 \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & \mathtt{x}_2 \\ \mathtt{x}_2 & \mathtt{w}_2 \end{bmatrix} \succeq 0 \right\}$$

in $\mathbb{R}^{2+2}$ which is represented by a LMI.

Let

$$S = \{\mathtt{x} \in \mathbb{R}^g : \rho_1(\mathtt{x}) \geq 0, \ldots, \rho_m(\mathtt{x}) \geq 0\} \qquad (24)$$

be a set defined by multivariate polynomials $\rho_i(\mathtt{x})$. Recall closed semialgebraic sets are finite unions of such sets. Since a compact LMI representable set is both convex and semialgebraic and the projection of a semialgebraic set is also semialgebraic (by the Tarski-Seidenberg principle), a SDP representable set must be convex and closed semialgebraic. What are sufficient conditions that guarantee S is SDP representable? Recently there has been substantial progress on this problem which we sketch here following the presentation in [19], [20].

A. SDP Representation Theorem

Assume S in (24) is convex, compact and has nonempty interior. Denote by ∂S the boundary of S. Let $Z_i = \{\mathtt{x} \in \mathbb{R}^g :$

$\rho_i(\mathtt{x}) = 0\}$ and note $\partial S \subset \cup_i Z_i$. We say the defining functions of S are nondegenerate provided $\nabla \rho_i(x) \neq 0$ for all $x \in Z_i \cap \partial S$. The boundary of S is said to have *positive curvature* provided that there exist nonsingular defining functions ρ_i for S such that at each $\mathtt{x} \in \partial S \cap Z_i$

$$-v^T \nabla^2 \rho_i(\mathtt{x}) v > 0, \quad \forall\, 0 \neq v \in \nabla \rho_i(\mathtt{x})^\perp \tag{25}$$

in other words, the Hessian of ρ_i compressed to the tangent space (the second fundamental form) is negative definite. A standard fact in geometry (follows from the chain rule) is that this does not depend on the choice of the defining function. The following is Theorem 3.9 of [20].

Theorem 6.1: Suppose S is a convex compact set with nonempty interior which has nondsingular defining polynomials $S = \{\mathtt{x} \in \mathbb{R}^g : \rho_1(\mathtt{x}) \geq 0, \ldots, \rho_m(\mathtt{x}) \geq 0\}$. If the boundary of S is positively curved, then S is SDP representable.

If S is convex with nonsingular defining functions, then its boundary has nonnegative curvature. Thus the positive curvature assumption is not a huge restriction beyond being convex.

Many variations of Theorem 6.1 are given in [20]. For example, the function ρ_i is called *sos-concave* provided $-\nabla^2 \rho_i(\mathtt{x}) = W(\mathtt{x})^T W(\mathtt{x})$ for some possibly nonsquare matrix polynomial $W(\mathtt{x})$. Then we can replace the curvature hypotheses on $Z_i \cap \partial S$ with the assumption that ρ_i is sos-concave.

B. Idea of Proof

The proof of the theorem is based on a semialgebraic geometry construction and then a variety of techniques are needed to validate the construction.

A construction of the SDP representation for convex sets was proposed by Lasserre [12] and also by Parrilo [36] and goes according to the following idea. Let $\mathcal{M}$ denote the space of Borel measures on S and let $\hat{S}$ denote the convex subset of all nonnegative mass one measures. Clearly, $\hat{S}$ projects down onto S via

$$P(\mu) := \int_S \mathtt{x} d\mu(\mathtt{x}) \qquad \mu \in \hat{S}.$$

Unfortunately $\hat{S}$ is infinite dimensional, so unsuitable as an SDP representation. The Lasserre and Parrilo proposal is to cut down $\hat{S}$ by looking at it as the set of all positive mass one linear functionals on the polynomials of some fixed degree N. Moment and sum of squares (SOS) techniques show that this gives a LMI, denoted by $\mathcal{L}_N$, for each degree N, and that the projection $P_{\mathbb{R}^g}\hat{S}$ onto $\mathtt{x}-$ space of the set $\hat{S}_N := \{(\mathtt{x}, \mathtt{y}) : \mathcal{L}_N(\mathtt{x}, \mathtt{y}) \geq 0\}$ contains S for all N. The open question remaining is whether there exists an integer N^* large enough to produce the equality. This is shown in [26] to hold if one can prove a Schmüdgen type Positivstellensätz with bounds on degrees of the unknown polynomials for certain perverse classes of functions. Parrilo [34] proved this construction gives a lifted LMI representation in the two dimensional case when the boundary of S is a single rational planar curve of genus zero. Lasserre [26] proved $P_{\mathbb{R}^g}\hat{S}$ converges to S when N goes to infinity.

The proof of Theorem 6.1 is based on validating a variation of this construction and requires several techniques of real algebraic geometry developed in [19], [20]. Of independent interest is the following theorem (which allows us to localize arguments).

Theorem 6.2: Suppose each convex set $W_j,\ j = 1, \ldots, k$ is SDP representable, is nonempty and is compact. Then the convex hull $\mathcal{K}$ of $\cup_j W_j$ has an SDP representation.

Apply this to to $W_j := S \cap B_j$ to obtain that $S = \textit{convex hull} \cup_j W_j$ is SDP representable, this helps prove Theorem 6.1.

VII. Algebraic Software

Here is a list of software running under NCAlgebra (which runs under Mathematica) that implements and experiments on symbolic algorithms pertaining to NC Convexity and LMIs.

http://www.math.ucsd.edu/~ncalg

- **LMI producing.** A symbolic algorithm of N. Slinglend has been implemented by J. Shopple under NCAlgebra to construct the linear pencil $\mathcal{L}$ in Theorem 3.3 symbolically from r. While it probably always associates a LMI with a convex r it works now only on small problems. Also to flexibly handle control problems more generality is needed and work is in progress. However, these results establish that the correspondence between dimension free systems problems which are convex and LMIs is very strong.
- **Convexity Checker.** Camino, Helton, Skelton have an (algebraic) algorithm for determining the region on which a rational expression is convex. See Section III-B.
- **Classical Production of LMIs.** There are two Mathematica NCAlgebra notebooks by de Oliveira and Helton. The first is based on algorithms for implementing the 1997 approach of Skelton, Iwasaki and Grigonidas [41] associating LMIs to more than a dozen control problems. The second (requires C++ and NCGB) produces LMIs by symbolically implementing the 1997 change of variables method of C. Scherer et. al.
- **Schur Complement Representations of an NC rational.** This computes a representation in (14) using the Shopple—Slinglend [40] algorithm. It is not known if p convex near 0 always leads to a monic pencil via this algorithm, but we never saw a counter example.
- **Determinantal Representations.** Finds Determinantal Representations of a given polynomial p. Shopple—Slinglend implements their [40] algorithm plus [18] algorithm. Requires NCAlgebra.

VIII. Mathematical Context and Background

The theme of this survey, namely the construction of an LMI from NC convex data, has as several main ingredients a few simple mathematical ideas influenced by positivity. Through its connection to positivity, the ubiquitous spectral theorem, which says that symmetric linear transformations are orthogonally diagonalizable, plays a great role.

More specifically, if $A = A^T$ is a linear bounded symmetric transformation of a (real) Hilbert space $\mathbf{H}$, and $p(\mathtt{x})$ is a real polynomial of a single variable, then

$$\|p(A)\| \leq \sup_{|\mathtt{x}| \leq \|A\|} |p(\mathtt{x})|.$$

This simple observation has far reaching consequences, including the full spectral decomposition theorem, and a more important possibly for our readers, the celebrated bounded analytic interpolation theorems, so much used and abused in H^∞-control theory. See [21] for an account of this line of thought, with almost complete proofs.

Even less expected is the fact the the spectral theorem (conveniently extended to commuting systems of symmetric operators) pays back to real algebra, and gives the following key Striktpositivstellensatz. We denote by $\mathbb{R}[\mathtt{x}]$ the algebra of real polynomials in the commuting variables $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_g)$ and by $\Sigma\mathbb{R}[\mathtt{x}]^2$ the convex cone of sums of squares of elements in $\mathbb{R}[\mathtt{x}]$. Let

$$S = \{\mathtt{x} \in \mathbb{R}^g;\ f_1(\mathtt{x}) \geq 0, \ldots, f_n(\mathtt{x}) \geq 0\}$$

be a basic semialgebraic set contained in the unit ball $||\mathtt{x}|| \leq 1$. Let $f_0(\mathtt{x}) = 1 - |\mathtt{x}|^2$.

Assume that a polynomial h satisfies $h(\mathtt{x}) > 0$ for all $\mathtt{x} \in S$. Then

$$h \in \Sigma\mathbb{R}[\mathtt{x}]^2 + f_0\Sigma\mathbb{R}[\mathtt{x}]^2 + \cdots + f_n\Sigma\mathbb{R}[\mathtt{x}]^2.$$

For an operator theory proof (based on elementary convexity theory and the spectral theorem), as well as for comments and bibliographical references on the main lines of a purely logico-algebraic proof see [21].

When dealing, as in the most part of the present survey, with tuples of matrices of arbitrarily large size, it is convenient to make a leap forward towards the abstract setting of free *-algebras. (Breaking with our convention, we both allow complex scalars and no longer assume that our variables are symmetric and hence we use *, rather than ^{T}, to denote the involution.) Specifically, we consider the algebra $\mathbb{R}\langle x, x^*\rangle$ where this time $x = (x_1, \ldots, x_g), x^* = (x_1^*, \ldots, x_g^*)$ are tuples of mutually free variables (i.e., no commutation relation are imposed). A linear involution $f \mapsto f^*$ acts on $\mathbb{R}\langle x, x^*\rangle$, so that

$$(fh)^* = h^*f^*, \quad (x_i)^* = x_i^*, (x_i^*)^* = x_i, \quad 1 \leq i \leq g.$$

We denote, to be consistent with the commutative picture, by $\Sigma\mathbb{R}\langle x,\ x^*\rangle^2$ the convex cone generated by elements of the form h^*h. As a matter of fact, all the computations sketched in the above sections (such as the proof that convex polynomials in matrix variables have degree two or less) take place in the algebra $\mathbb{R}\langle x, x^*\rangle$. Indeed, if $A = (A_1, \ldots, A_g)$ is a tuple of matrices, then

$$f(x, x^*) \mapsto f(A, A^T)$$

provides a representation of $\mathbb{R}\langle x, x^*\rangle$ into a finite dimensional matrix algebra containing A. The key Nichtnegativstellensätz (Theorem 4.1 above) reads then as:

Assume $h \in \mathbb{R}\langle x, x^*\rangle$ satisfies $h(A, A^T) \geq 0$ in all matrix representations. Then $h \in \Sigma\mathbb{R}\langle x, x^*\rangle^2$.

Again a convexity theory and a basic operator theory construction provides an accessible proof of this result and a variety of variants of it, with supports. (See [21].)

A fascinating direction of research, well on its way at the abstract level, but not yet absorbed by the applied mathematical community, is to consider Positivstellensätze in algebras with involution which are intermediate between $\mathbb{R}[\mathtt{x}]$ and $\mathbb{R}\langle x, x^*\rangle$, as for instance Weyl's algebra of linear differential operators with polynomial coefficients, enveloping algebras of Lie algebras, matrix algebras. For examples and typical results see [13], [23], [37]–[39].

In our opinion, of immediate importance for the engineering community, is the mixed algebra in commuting variables $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_g)$ and noncommuting ones $x = (x_1, \ldots, x_{g'})$. A basic Positivstellensatz in this case can be derived along the lines exposed in [21].

The message of this short section is: even if in the present survey we have interchanged without restrictions the terms "noncommuting variables" and "matrix variables", mathematically we meant that we do the computations at the free-* algebra level, and descend with the final conclusion standing at the lower (and much more involved) level of polynomials or matrices of a specific degree, respectively dimension. As often in our discipline, the higher the level of abstraction, the easier the proofs are.

IX. Conclusion

Commutative: Semialgebraic geometry is a subject which goes back 75 to 100 years.

- Conjecture: rigidly convex semialgebraic sets are precisely those with a LMI representation. Rigid convexity is considerable stronger than convexity. (See Section V.)
- Convex basic semialgebraic sets with nonsingular positively curved boundary lift to a convex set having a LMI representation. (See Section VI.)
- Change of variables to make a problem convex is classically analyzed (via Morse theory) and understood, but what one finds seems not to be profoundly practical.

Noncommutative: NC semialgebraic geometry (in our acceptation of the term) is only a few years old and, while challenging, is developing well into a mathematically rich area.

- Conjecture: NC convexity is the same as having an NCLMI representation. While far from proved, evidence for that conclusion is strong. (See Sections II-C–III-A.)
- Changing variables to make a NC problem convex is an open area. It is possible that NC changes of variables will behave better than classically.

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