PARABOLIC GEOMETRIC EISENSTEIN SERIES AND CONSTANT TERM FUNCTORS

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Abstract. We prove a compatibility between parabolic restriction of Whittaker sheaves and restriction of representations under the geometric Casselman-Shalika equivalence. To do this, we establish various Hecke structures on geometric Eisenstein series functors, generalizing results of Braverman-Gaitsgory in the case of a principal parabolic. Moreover, we relate compactified and non-compactified geometric Eisenstein series functors via Koszul duality.

We sketch a proof that the spectral-to-automorphic geometric Langlands functor commutes with constant term functors.

Contents

1. Introduction

1.1. **Framework.** Let G be a reductive connected algebraic group over a field k of characteristic zero. Fix a parabolic subgroup $P \subseteq G$ with Levi quotient M and unipotent radical N_P. Let $\check{G}$ be the Langlands dual reductive group of G with corresponding parabolic subgroup $\check{P} = \check{M}\check{N}_P$. We fix a smooth projective curve X over k.

1.1.1. This paper is concerned with geometric Eisenstein series, which we presently recall. By induction of princial bundles along the correspondence $M \leftarrow P \rightarrow G$, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Bun}_P & & \\ & \overset{q}{\swarrow} & \downarrow j & \overset{p}{\searrow} & \\ & & \widetilde{\mathrm{Bun}}_P & & \\ & \overset{\widetilde{q}}{\swarrow} & & \overset{\widetilde{p}}{\searrow} & \\ \mathrm{Bun}_M & & & & \mathrm{Bun}_G, \end{array} \tag{1.1.1}$$

where for an algebraic group H, Bun_H is the moduli stack of principal H-bundles over X. The algebraic stack $\widetilde{\mathrm{Bun}}_P$ denotes Drinfeld's relative compactification of p.

1.1.2. For or a prestack $\mathcal{Y}$, let $D(\mathcal{Y})$ be the category of D-modules on $\mathcal{Y}$. Pull-push along the diagram (1.1.1) provides us with Eisenstein series functors, the protagonists of this paper. More precisely, we have the functors:

$$\mathrm{Eis}_! : D(\mathrm{Bun}_M) \to D(\mathrm{Bun}_G), \;\; \mathcal{F} \mapsto p_* \circ q^!(\mathcal{F});$$

$$\mathrm{Eis}_{!*} : D(\mathrm{Bun}_M) \to D(\mathrm{Bun}_G), \;\; \mathcal{F} \mapsto \widetilde{p}_*(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \overset{!}{\otimes} \widetilde{q}^!(\mathcal{F})),$$

which we call (geometric) *Eisenstein series* and *compactified Eisenstein series*, respectively.

1.1.3. We will also consider several local variations of the functors $\mathrm{Eis}_!$ and $\mathrm{Eis}_{!*}$ as well as their dual counterparts, the *constant term functors*.[1]

The goal of this paper is to establish some fundamental properties of geometric Eisenstein series and constant term functors. More precisely:

[1]The name *constant term functor* is sometimes used in the literature to denote an adjoint functor to Eisenstein series and sometimes to denote its dual functor. We allow ourselves to be vague for now on which of the two meanings we are using.

(1) We establish a certain compatibility between geometric Eisenstein series and Hecke functors. This generalizes a result of Braverman-Gaitsgory [BG99], [BG06] in the case of a principal parabolic.
(2) We prove that restriction of representations along $\check{M} \to \check{G}$ goes to a suitable Jacquet functor on the spherical Whittaker category under the geometric Casselman-Shalika equivalence.
(3) As a corollary to the second point above, we characterize the Koszul dual Jacquet functor as invariants with respect to the Lie algebra $\check{\mathfrak{n}}_P$ of the unipotent radical of $\check{P}$. This generalizes a result of Raskin [Ras21] in the principal case.

1.1.4. *Relation to the proof of the geometric Langlands conjecture.* A fundamental step in the proof of the geometric Langlands conjecture is to establish that the Langlands functor

$$\mathbb{L}_G : D(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})$$

interchanges geometric and spectral Eisenstein series, suitably understood. This was proven in [CCF+24]. The arguments use in an essential way results of Campbell-Raskin [CR23] who establish a semi-infinite geometric Satake theorem. However, we wish to emphasize that the contents of the paper [CR23] use our results stated in point (3) above in an essential way, and in particular, neither [CR23] nor [CCF+24] should be viewed as implying the theorems established in the present text.

1.2. **Some context.** To set the stage of our main results, let us review the results of Braverman-Gaitsgory [BG99], [BG06] on the compatibility between Eisenstein series and Hecke functors mentioned in point (1) of §1.1.3. They work under the assumption that the parabolic P is a Borel subgroup B. In this case, we denote Drinfeld's compactification by $\overline{\mathrm{Bun}}_B$.

1.2.1. The basic results can be summarized as follows, although we refer to [FM97],[FFKM97], [BFGM02],[BG99],[BG06] for more details.

Let Conf be the space of divisors on X valued in the monoid Λ^{pos} of positive linear combinations of positive roots in G. One has two factorization algebras $\mathcal{O}(\check{N})_{\mathrm{Conf}}, \Omega(\check{\mathfrak{n}})_{\mathrm{Conf}} \in D(\mathrm{Conf})$ obtained from the coalgebra $\mathcal{O}(\check{N})$ of functions on $\check{N}$ and the algebra $C^\bullet(\check{\mathfrak{n}})$ given by the cohomological Chevalley complex of $\check{\mathfrak{n}}_P$, respectively. We have actions:

$$\mathcal{O}(\check{N})_{\mathrm{Conf}} \curvearrowright \mathrm{IC}_{\overline{\mathrm{Bun}}_B}, \;\; \Omega(\check{\mathfrak{n}})_{\mathrm{Conf}} \curvearrowright j_!(\omega_{\mathrm{Bun}_B}).$$

The usual Koszul duality between $\mathcal{O}(\check{N})$ and $C^\bullet(\check{\mathfrak{n}})$ provides a 'Koszul duality' between $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ and $j_!(\omega_{\mathrm{Bun}_B})$. That is, taking invariants of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ as a module sheaf for $\mathcal{O}(\check{N})_{\mathrm{Conf}}$ yields $j_!(\omega_{\mathrm{Bun}_B})$, and vice versa. This similarly shows that the functors given by the kernels $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}, j_!(\omega_{\mathrm{Bun}_B})$, namely $\mathrm{Eis}_{!*}$ and $\mathrm{Eis}_!$, respectively, are 'Koszul dual'.

We have an equivalence:

$$\Omega(\check{\mathfrak{n}})_{\mathrm{Conf}}\text{-mod}(D(\mathrm{Bun}_T)) \simeq D(\mathrm{Bun}_T) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{T}})}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}).$$

This shows that the functor $\mathrm{Eis}_!$ factors through a functor:

$$\mathrm{Eis}_!^{\mathrm{enh}} : D(\mathrm{Bun}_T) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{T}})}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}) \to D(\mathrm{Bun}_G).$$

In the language introduced in this paper, this reflects the fact that $j_!(\omega_{\mathrm{Bun}_B})$ is equipped with an *enhanced Drinfeld-Plücker structure*, see §1.5.4 below.

1.3. **Why are we writing this paper?**

1.3.1. The proofs of the results stated in §1.2.1 required a tremendous amount of control of the singularities of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. The above actions ultimately come from describing how $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ decomposes when restricted to a natural stratification of $\overline{\mathrm{Bun}}_B$, and verifying that the action maps are compatible with the (co)algebra structures is a very subtle affair.

While one of the main motivations for writing this paper is to generalize the results stated in §1.2.1 from the principal case to a general parabolic, another major motivation is to provide a short, self-contained proof of these results that completely avoids the just-mentioned subtleties.

1.3.2. Our starting point is a certain local analogue of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ known as the *semi-infinite intersection cohomology sheaf* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$. Its definition, fundamental properties, and local-to-global compatibilities in the case when $P = B$ is a Borel were first established by Gaitsgory in [Gai18], [Gai21].

Remark 1.3.2.1. While writing this paper, a definition of the fiber $\mathrm{IC}^{\frac{\infty}{2}}_{P,x}$ of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ at a point x of X appeared in [DL25]. It is our understanding that Dhillon-Lysenko plan to release a second paper studying $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ factorizably.

Another definition of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ as a factorization algebra also appears in the paper [Hay25] of the second author, using Zastava spaces. It should be noted that the methods used in the present paper differ from the above references in that we use Hecke structures to define $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ rather than employing colimit constructions or t-structures directly.

In more concrete terms, $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is a sheaf on a suitable prestack

$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \widetilde{\mathrm{Bun}}_P, \tag{1.3.1}$$

see Section 5 for its definition. The semi-infinite IC-sheaf is designed to satisfy many of the properties required of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to establish the results in §1.2.1. As such, instead of starting with $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ and proving its desired properties, one starts with $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$, where these properties are much more evident, and then identifies the latter with $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ under the map (1.3.1). The upshot is that it is often easy to identify a given sheaf with an IC-sheaf: one simply has to control the perverse degrees of the ! and $*$-restrictions to the singular locus as well as its restriction to the smooth locus.

1.3.3. In fact, one can replace $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ with the pushforward of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ along the map (1.3.1) in the definition of $\mathrm{Eis}_{!*}$ and not lose any information.[2] It is our understanding that this approach will be taken by Hamann-Hansen-Scholze in their work[3] on geometric Eisenstein series on stacks of bundles on the Fargues-Fontaine curve; the point being that in this setting, there is no perverse t-structure.

1.3.4. The other main purpose of this paper is to prove a certain compatibility between geometric and spectral parabolic restriction under the geometric Casselman-Shalika equivalence (see §1.4 below for a precise statement). This generalizes a result of Raskin in the principal case [Ras21].

Given this result, we sketch how to prove that the spectral-to-automorphic Langlands functor

$$\mathbb{L}_G^{\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to D(\mathrm{Bun}_G)$$

[2]For example, in principle one never needs to consider $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ to prove that $\mathbb{L}_G$ interchanges the two Eisenstein series functors.

[3]The work we allude to here is the second of their series of papers on geometric Eisenstein series and is unavailable at the time of writing. See, however, the first paper in the series: [HHS24].

commutes with constant term functors. One should consider this equivalent to the assertion that the automorphic-to-spectral Langlands functor $\mathbb{L}_G$ commutes with (suitable) Eisenstein series functors, see §1.4.5 below for more details.

1.3.5. Finally, we note that the results in this paper have been cited in the literature already. Besides the relation to the work of Campbell-Raskin mentioned in §1.1.4, the Hecke structure on geometric Eisenstein series for an arbitrary parabolic (Theorem 1.6.5.2 below) played a key role in many important results in geometric Langlands, such as:

- The spectral decomposition of $D(\mathrm{Bun}_G)$ over $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ [Gai10].
- The existence of Hecke eigensheaves associated to irreducible local systems (a folklore result written up in [FR22, §11]).
- (A more recent application:) combining the characterization of nilpotent singular support in terms of Hecke functors given in [AGK+20] with the Hecke structure on geometric Eisenstein series proves that geometric Eisenstein series preserves nilpotent singular support.[4] In the ℓ-adic setting, this is a crucial input if one wants to study pseudo-Eisenstein series of automorphic forms via geometric Eisenstein series through categorical trace of Frobenius (see [Ras24] for one such example).

1.4. **Jacquet Functors.** In the next two subsections, we describe the main results in this paper in more detail.

1.4.1. *Semi-infinite IC-sheaf.* The main player in the local story is the parabolic semi-infinite IC-sheaf:

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \in D(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

The first basic properties of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ are:

- $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ naturally belongs to the *semi-infinite category* $\mathrm{SI}_{P,\mathrm{Ran}} := D(\mathfrak{L}_{\mathrm{Ran}}N_P\mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$.
- Denote by $\widetilde{S}^0_{P,\mathrm{Ran}} \subseteq \mathrm{Gr}_{G,\mathrm{Ran}}$ the closure of the 0'th semi-infinite $\mathfrak{L}_{\mathrm{Ran}}N_P$-orbit (see §4.4.2 for a precise definition). Then $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is supported on $\widetilde{S}^0_{P,\mathrm{Ran}}$.

1.4.2. Define

$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} := \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash\widetilde{S}^0_{P,\mathrm{Ran}};$$

$$\mathrm{Gr}_{P,\mathrm{Ran}} := \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash S^0_{P,\mathrm{Ran}}.$$

We have an open embedding:

$$j_{\mathrm{Ran}} : \mathrm{Gr}_{P,\mathrm{Ran}} \hookrightarrow \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}.$$

For convenience, write $j_{\mathrm{Ran},!} := j_{\mathrm{Ran},!}(\omega_{\mathrm{Gr}_{P,\mathrm{Ran}}})$.

Consider the diagram:

$$\begin{array}{ccccc} & & \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} & \xrightarrow{\widetilde{p}_{\mathrm{Ran}}} & \mathrm{Gr}_{G,\mathrm{Ran}} \\ & \swarrow^{\widetilde{q}_{\mathrm{Ran}}} & & \searrow^{\pi} & \\ \mathrm{Gr}_{M,\mathrm{Ran}} & & & & \mathfrak{L}^+_{\mathrm{Ran}}M\backslash\widetilde{S}^0_{P,\mathrm{Ran}}. \end{array}$$

[4] In forthcoming joint work with Marius Kjærsgaard, the first author establishes a purely geometric proof that geometric Eisenstein series and constant term functors preserve nilpotent singular support, independent of the current paper and [AGK+20].

From here, we may consider the functors:

$$\mathrm{Jac}^M_! : D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to D(\mathrm{Gr}_{M,\mathrm{Ran}}), \;\; \mathcal{F} \mapsto \widetilde{q}_{\mathrm{Ran},*}(j_{\mathrm{Ran},!} \overset{!}{\otimes} \widetilde{p}^!_{\mathrm{Ran}}(\mathcal{F}));$$
$$\mathrm{Jac}^M_{!*} : D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to D(\mathrm{Gr}_{M,\mathrm{Ran}}), \;\; \mathcal{F} \mapsto \widetilde{q}_{\mathrm{Ran},*}(\pi^!(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \overset{!}{\otimes} \widetilde{p}^!_{\mathrm{Ran}}(\mathcal{F})).$$

These functors descend to functors on the Whittaker categories that we similarly denote by $\mathrm{Jac}^M_!$ and $\mathrm{Jac}^M_{!*}$:

$$\mathrm{Jac}^M_! : \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}}));$$
$$\mathrm{Jac}^M_{!*} : \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) \to \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}})).$$

Here, the Whittaker conditions for G and M are understood with respect to a non-degenerate character of $\mathfrak{L}_{\mathrm{Ran}}N$ and its restriction to $\mathfrak{L}_{\mathrm{Ran}}N_M$, respectively, see Section 6.

1.4.3. Recall that the (factorizable) geometric Casselman-Shalika formula provides an equivalence

$$\mathrm{CS}_G : \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) \simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

of factorization categories. On the Langlands dual side, we may consider the functors

$$C^\bullet(\check{\mathfrak{n}}_P, -) : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}};$$
$$\mathrm{Res}^{\check{G}}_{\check{M}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

induced by taking Lie algebra cohomology along $\check{\mathfrak{n}}_P = \mathrm{Lie}(\check{N}_P)$ and restricting along $\check{M} \to \check{G}$, respectively.

Our main result is the following:

Theorem 1.4.3.1 (Corollary 6.4.1.2, Theorem 6.4.1.1)**.** *We have commutative diagrams:*

$$\begin{array}{ccc} \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) & \xrightarrow{\mathrm{Jac}^M_!} & \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}}) \\ {\scriptstyle \mathrm{CS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{CS}_M} \\ \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow[C^\bullet(\check{\mathfrak{n}}_P,-)]{} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}}, \end{array}$$

$$\begin{array}{ccc} \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) & \xrightarrow{\mathrm{Jac}^M_{!*}} & \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}})) \\ {\scriptstyle \mathrm{CS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{CS}_M} \\ \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow[\mathrm{Res}^{\check{G}}_{\check{M}}]{} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}}. \end{array}$$

1.4.4. The commutativity of the first diagram was proved when $P = B$ is a Borel subgroup by Raskin in [Ras21].[5] The proof in *loc.cit* relies on results of [BG06]. Our proof of Theorem 1.4.3.1 is self-contained except for using a certain vanishing result of [Ras21, §3], the latter result being independent from the rest of [Ras21]. We also sketch an argument for how to circumvent the vanishing result of Raskin, see §6.4.

[5]In truth, Raskin proves that the functor $\mathrm{Jac}^M_* : \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) \to \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}}))$ defined by replacing $j_{\mathrm{Ran},!}$ with $j_{\mathrm{Ran},*}$ in the definition of $\mathrm{Jac}^M_!$ corresponds to taking Lie algebra *ho*mology on the Langlands dual side.

1.4.5. $\mathbb{L}_G$ *interchanges Eisenstein series.* Finally, let us illustrate the importance of Theorem 1.4.3.1 by explaining its relevance to the proof in [CCF+24] that the geometric Langlands functor

$$\mathbb{L}_G : D(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})$$

interchanges Eisenstein series.[6] We do this to orient the reader about how the overall logic of the proof in [CCF+24] goes and in particular to highlight that Theorem 1.4.3.1 is *the* place one has to deal with Langlands duality.

For simplicity, we ignore issues about temperedness, twists, translations, Cartan involution and shifts in what follows.

There is a natural localization functor

$$\mathrm{Loc}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

with a fully faithful right adjoint $\mathrm{coLoc}_{\check{G}}$. By the vanishing result of [Gai10], the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ by Hecke functors factors through an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. By acting on the Whittaker sheaf, we obtain a functor:

$$\mathbb{L}_G^{\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D(\mathrm{Bun}_G).$$

Similarly, we have a Poincaré functor

$$\mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) \xrightarrow{\mathrm{Poinc}_{G,!}} D(\mathrm{Bun}_G)$$

that fits into a commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G^{-1}} & \mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) \\ \downarrow{\scriptstyle \mathrm{Loc}_{\check{G}}} & & \downarrow{\scriptstyle \mathrm{Poinc}_{G,!}} \\ \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow[\mathbb{L}_G^{\mathrm{spec}}]{} & D(\mathrm{Bun}_G). \end{array}$$

It suffices to show that the diagram

$$\begin{array}{ccc} D(\mathrm{Bun}_M) & \xleftarrow{\mathbb{L}_M^{\mathrm{spec}}} & \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \\ \uparrow{\scriptstyle \mathrm{CT}_{P,!}} & & \uparrow{\scriptstyle \mathrm{CT}_P^{\mathrm{spec}}} \\ D(\mathrm{Bun}_G) & \xleftarrow[\mathbb{L}_G^{\mathrm{spec}}]{} & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \end{array} \qquad (1.4.1)$$

commutes. Here, $\mathrm{CT}_{P,!}$ is the functor of $*$-pull, !-push along:

$$\mathrm{Bun}_M \leftarrow \mathrm{Bun}_P \rightarrow \mathrm{Bun}_G,$$

and $\mathrm{CT}_P^{\mathrm{spec}}$ is the functor of pull-push along

$$\mathrm{LocSys}_{\check{M}} \leftarrow \mathrm{LocSys}_{\check{P}} \rightarrow \mathrm{LocSys}_{\check{G}}.$$

The reader should think that the implication

$$(\text{commutativity of (1.4.1)}) \to (\mathbb{L}_G \text{ interchanges Eisenstein series})$$

[6]We remark that *loc.cit* also establishes that $\mathbb{L}_G$ commutes with constant term functors, an assertion we do not address at all in this paper.

follows by taking dual functors in (1.4.1), where we identify $D(\mathrm{Bun}_G)$ and $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ with their own duals through miraculous duality for the former and in the standard way for the latter.[7]

Let us expand (1.4.1) to the following diagram:

$$\begin{CD}
D(\mathrm{Bun}_M) @<{\mathbb{L}_M^{\mathrm{spec}}}<< \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \\
@A{\mathrm{CT}_{P,!}}AA @AA{\mathrm{CT}_P^{\mathrm{spec}}}A \\
D(\mathrm{Bun}_G) @<<{\mathbb{L}_G^{\mathrm{spec}}}< \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \\
@A{\mathrm{Poinc}_{G,!}}AA @AA{\mathrm{Loc}_{\check{G}}}A \\
\mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) @<<{\mathrm{CS}_G^{-1}}< \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.
\end{CD} \tag{1.4.2}$$

We already noted that the lower diagram commutes. Thus, it suffices to show that the outer diagram commutes.

We have commutative diagrams:

$$\begin{CD}
\mathrm{Whit}(D(\mathrm{Gr}_{G,\mathrm{Ran}})) @>{\mathrm{Jac}_!^M}>> \mathrm{Whit}(D(\mathrm{Gr}_{M,\mathrm{Ran}})) \\
@V{\mathrm{Poinc}_{G,!}}VV @VV{\mathrm{Poinc}_{M,!}}V \\
D(\mathrm{Bun}_G) @>>{\mathrm{CT}_{P,!}}> D(\mathrm{Bun}_M),
\end{CD}
\qquad
\begin{CD}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} @>{C^\bullet(\check{\mathfrak{n}}_P,-)}>> \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \\
@V{\mathrm{Loc}_{\check{G}}}VV @VV{\mathrm{Loc}_{\check{M}}}V \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) @>>{\mathrm{CT}_P^{\mathrm{spec}}}> \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}).
\end{CD}$$

The commutativity of the left diagram was established in [Lin23], and the commutativity of the right diagram was established in [CCF+24, Thm. 12.2.10].[8] Putting it all together, the commutativity of the outer diagram of (1.4.2) follows from our Theorem 1.4.3.1.

Remark 1.4.5.1. We remark here that the commutativity of the right diagram can be proved roughly as follows. The diagram

$$\begin{CD}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} @>{\mathrm{Res}_{\check{M}}^{\check{G}}}>> \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \\
@V{\mathrm{Loc}_{\check{G}}}VV @VV{\mathrm{Loc}_{\check{M}}}V \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) @>>{\iota^*}> \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})
\end{CD}$$

commutes, where $\iota : \mathrm{LS}_{\check{M}} \to \mathrm{LS}_{\check{G}}$ denotes the map of inducing local systems along $\check{M} \to \check{G}$, and pullback along this map is obviously compatible with the evident $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-actions. Here we view $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ as a monoidal category under *external* convolution. Both functors carry actions of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ (see §1.5 below for the definition of the latter). Taking invariants for $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$, we obtain a commutative diagram with target category $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}))$.

[7]We have grossly simplified this implication step. As a first point, the results of [Lin23] are needed to make this precise.

[8]Really one has to precompose $\mathrm{Jac}_!^M$ and $C^\bullet(\check{\mathfrak{n}}_P, -)$ with the functor of inserting the unit (cf. [ABC+24, §C.11]), but we ignore this here.

Finally, one can show that

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}),$$

and that under this equivalence, the functor $\mathrm{CT}_P^{\mathrm{spec}}$ corresponds to the composition

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \longrightarrow \Omega(\check{\mathfrak{n}}_P)\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})) \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}),$$

where the second arrow is the forgetful functor.[9]

1.5. **Hecke structures and Koszul duality.** The second main result of this paper is the construction of Hecke and Drinfeld-Plücker structures on $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ and related sheaves. We refer to Section 4 for more details.

1.5.1. *Koszul duality.* Consider $\mathcal{O}(\check{N}_P)$ as a coalgebra object of $\mathrm{Rep}(\check{M})$. Similarly, consider $C^\bullet(\check{\mathfrak{n}}_P)$, the cohomological Chevalley complex of $\check{\mathfrak{n}}_P$, as an algebra object of $\mathrm{Rep}(\check{M})$. We define (co)algebra objects

$$\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}, \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

whose !-fibers at some $x \in X$ recover $\mathcal{O}(\check{N}_P), C^\bullet(\check{\mathfrak{n}}_P)$, respectively.

1.5.2. Consider the semi-infinite category

$$\mathrm{SI}_{P,\mathrm{Ran}} = D(\mathfrak{L}_{\mathrm{Ran}} N_P \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$

Note that $\mathrm{SI}_{P,\mathrm{Ran}}$ is naturally a module category for $\mathrm{Rep}(\check{M} \times \check{G})_{\mathrm{Ran}}$. As mentioned in §1.4.1, the semi-infinite IC-sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ defines an object of $\mathrm{SI}_{P,\mathrm{Ran}}$.

Koszul duality provides an equivalence of categories

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}} : \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathrm{SI}_{P,\mathrm{Ran}}) \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{SI}_{P,\mathrm{Ran}})$$

given by taking invariants for $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ (see Lemma 4.2.7.1 for a general statement of this type).

1.5.3. Denote by $\mathbf{j}_!$ the !-pushforward of the dualizing sheaf along the open embedding $S^0_{P,\mathrm{Ran}} \hookrightarrow \widetilde{S}^0_{P,\mathrm{Ran}}$. We have:

Theorem 1.5.3.1 (Proposition 4.5.4.1)**.** *The sheaf* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ *is equipped with a canonical comodule structure for* $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$. *Moreover, we have:*

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq \mathbf{j}_!.$$

In particular, $\mathbf{j}_!$ *is equipped with a module structure for* $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$.

1.5.4. *Hecke structures.* Let $\overline{\check{N}_P\backslash\check{G}}$ be the affinization of $\check{N}_P\backslash\check{G}$.

For a scheme X, we denote by $\mathcal{O}(X)$ the derived global section of the structure sheaf of X. We consider the schemes

$$\check{G},\ \check{N}_P\backslash\check{G},\ \overline{\check{N}_P\backslash\check{G}}$$

as acted on by $\check{M} \times \check{G}$ in the obvious ways. We get algebras

$$\mathcal{O}(\check{G})_{\mathrm{Ran}},\ \mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}},\ \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M} \times \check{G})_{\mathrm{Ran}}$$

whose fiber at a point $x \in X$ recovers $\mathcal{O}(\check{G}), \mathcal{O}(\check{N}_P\backslash\check{G}), \mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \in \mathrm{Rep}(\check{M} \times \check{G})$, respectively.

[9]We have ignored one subtlety: taking invariants for $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ as an algebra object in $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ with *external* convolution does not recover the functor $C^\bullet(\check{\mathfrak{n}}_P, -)$. Rather, this functor is given by taking invariants when $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ is considered an algebra object in $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ with *pointwise* convolution. However, these differ exactly by the functor of inserting the unit, cf. the previous footnote.

1.5.5. We define three categories consisting of objects of $\mathrm{SI}_{P,\mathrm{Ran}}$ equipped with extra structure related to the action of $\mathrm{Rep}(\check{M}\times\check{G})_{\mathrm{Ran}}$:

(1) $\mathrm{Hecke}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}}) := \mathcal{O}(\check{G})_{\mathrm{Ran}}\text{-mod}(\mathrm{SI}_{P,\mathrm{Ran}})$.
For $\mathcal{F}\in\mathrm{SI}_{P,\mathrm{Ran}}$, we refer to a lift of $\mathcal{F}$ to $\mathrm{Hecke}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}})$ as a *Hecke structure* on $\mathcal{F}$. Concretely, this amounts to a family of isomorphisms:
$$\mathcal{F}\star V \simeq \mathrm{Res}^{\check{G}}_{\check{M}}(V)\star\mathcal{F},\ V\in\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$
satisfying natural higher compatibilities.

(2) $\mathrm{DrPl}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}}) := \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}\text{-mod}(\mathrm{SI}_{P,\mathrm{Ran}})$.
For $\mathcal{F}\in\mathrm{SI}_{P,\mathrm{Ran}}$, we refer to a lift of $\mathcal{F}$ to $\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}})$ as a *Drinfeld-Plücker structure* on $\mathcal{F}$. At a point $x\in X$, this amounts to a family of maps:
$$V^{\check{N}_P}\star\mathcal{F}\to\mathcal{F}\star V,\ V\in\mathrm{Rep}(\check{G})$$
satisfying natural higher compatibilities. Here $V^{\check{N}_P}$ is the *underived* invariants of V along $\check{N}_P$.

(3) $\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}}) := \mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}}\text{-mod}(\mathrm{SI}_{P,\mathrm{Ran}})$.
For $\mathcal{F}\in\mathrm{SI}_{P,\mathrm{Ran}}$, we refer to a lift of $\mathcal{F}$ to $\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}})$ as an *enhanced Drinfeld-Plücker structure* on $\mathcal{F}$. Concretely, this amounts to a family of isomorphisms:
$$\mathcal{F}\star V\simeq C^\bullet(\check{\mathfrak{n}}_P,V)\underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star}\mathcal{F},\ V\in\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$
satisfying natural higher compatibilities.

Theorem 1.5.5.1. $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ *is canonically equipped with a Hecke structure. Moreover,* $\mathbf{j}_!$ *is canonically equipped with an enhanced Drinfeld-Plücker structure.*

Remark 1.5.5.2. In fact, contrary to [Gai18][Gai21], we *define* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ by the requirement that it possesses a Hecke structure. Since we have avoided defining $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ in this introduction, we keep the first part of the above result as a theorem for simplicity. On the other hand, that $\mathbf{j}_!$ possesses an enhanced Drinfeld-Plücker structure is quite non-trivial and follows from Theorem 1.5.3.1.

This illustrates one of the main points of this paper rather well: instead of trying to define the enhanced Drinfeld-Plücker structure on $\mathbf{j}_!$ directly, we define another sheaf[10] with an enhanced Drinfeld-Plücker structure and identify it with $\mathbf{j}_!$.[11] If, for example, one tried to construct isomorphisms
$$\mathbf{j}_!\star V\simeq C^\bullet(\check{\mathfrak{n}}_P,V)\underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star}\mathbf{j}_!,\ V\in\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$
directly, this would be very subtle.

1.5.6. *Where do these structures come from?* We have highlighted the advantage of our approach to constructing, say, Hecke structures on sheaves of interest compared to the approach taken previously in the literature, as this required constructing essentially all the structures 'by hand'. This begs the question of whether we have completely removed the necessity of doing things by hand.

This is not quite true. After all, our statements involve Langlands duality,[12] and so combinatorics has to appear at some point. In our case, this appears when having to construct a Drinfeld-Plücker

[10]In this case $\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$.

[11]Which is in principle straightforward: to give an isomorphism $\mathcal{F}\simeq\mathbf{j}_!$, one has to identify $\mathcal{F}$ with the dualizing sheaf upon restriction to $S^0_{P,\mathrm{Ran}}$ and show that the $*$-restriction of $\mathcal{F}$ to $\widetilde{S}^0_{P,\mathrm{Ran}}\backslash S^0_{P,\mathrm{Ran}}$ vanishes.

[12]Langlands duality that does not come from geometric Satake, that is.

structure on the delta sheaf along the unit section $\mathrm{Ran} \to \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}$ from which we bootstrap everything. However, constructing this Drinfeld-Plücker structure is very simple, and we regard it as the minimal 'combinatorical' input needed to obtain the results of this paper.

1.6. **Local-to-global comparisons.** In this subsection, we describe how to relate the semi-infinite IC-sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to the IC-sheaf of $\widetilde{\mathrm{Bun}}_P$. We refer to Section 5 for more details.

1.6.1. We let
$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \mathrm{Bun}_M$$
denote a relative version of $\widetilde{S}^0_{P,\mathrm{Ran}}$ living over Bun_M, see §5.1. By definition, it comes with a canonical map:
$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \widetilde{S}^0_{P,\mathrm{Ran}}. \tag{1.6.1}$$

We let ${}_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}, {}_{\mathrm{Bun}_M}\mathbf{j}_!$ denote the !-pullbacks of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}, \mathbf{j}_!$, respectively.

1.6.2. We have a natural map:
$$\pi_P : \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \widetilde{\mathrm{Bun}}_P.$$

Theorem 1.6.2.1 (Theorem 5.3.8.1)**.** *We have a canonical isomorphism:*
$$\pi_P^!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})[\dim(\mathrm{Bun}_P)] \simeq \mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}.$$
Moreover, the counit map
$$\pi_{P,!} \circ \pi_P^!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \to \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$$
is an isomorphism.

Remark 1.6.2.2. Here, $\dim(\mathrm{Bun}_P)$ denotes the locally constant function on Bun_P that gives the dimension of a given connected component.

1.6.3. Consider the prestack $\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$ over Ran that parametrizes a point $x_I \in \mathrm{Ran}$ and a generalized P-reduction on X that is non-degenerate away from x_I (see §5.2.3 for a precise definition).

The main property of $\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$ is that it carries Hecke modifications for both M and G. That is, the category $D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}})$ is a module category for $\mathrm{Rep}(\check{M} \times \check{G})_{\mathrm{Ran}}$. This allows us to talk about Hecke structures and (enhanced) Drinfeld-Plücker structures of objects of $D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}})$ as in §1.5.5.

1.6.4. We have an open embedding:
$$j_{P,\mathrm{pol}} : \mathrm{Bun}_P \times \mathrm{Ran} \to \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}.$$
We define $\mathbf{j}^{\mathrm{glob}}_! := j_{P,\mathrm{pol},!}(\omega_{\mathrm{Bun}_P \times \mathrm{Ran}}) \in D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}})$.

1.6.5. We have a closed substack
$$i_{P,\mathrm{pol}} : \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \hookrightarrow \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$$
defined by requiring the Drinfeld-Plücker maps be regular. It comes with a tautological map forgetting the point of Ran:
$$\mathrm{oblv}_{\mathrm{zer}} : \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \to \widetilde{\mathrm{Bun}}_P.$$

We define
$$\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}} := i_{P,\mathrm{pol},*} \circ \mathrm{oblv}^!_{\mathrm{zer}}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \in D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}).$$
As a consequence of Theorem 1.6.2.1 and the results in §1.5, we obtain:

Theorem 1.6.5.1. $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}$ *is a equipped with a canonical comodule structure for* $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$. *Moreover, we have:*

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}) \simeq \mathbf{j}_!^{\mathrm{glob}}.$$

In particular, $\mathbf{j}_!$ *is equipped with a module structure for* $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$.

Theorem 1.6.5.2. $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}$ *is canonically equipped with a Hecke structure. Moreover,* $\mathbf{j}_!^{\mathrm{glob}}$ *is canonically equipped with an enhanced Drinfeld-Plücker structure.*

1.7. **Organization of the paper.** In Section 2, we recall notations and conventions.

In Section 3, we introduce the main geometric players that appear in this paper.

In Section 4, we introduce and study the parabolic semi-infinite IC-sheaf.

In Section 5, we prove the local-to-global comparison between the semi-infinite IC-sheaf and the IC-sheaf on $\widetilde{\mathrm{Bun}}_P$.

In Section 6, we prove the compatibility stated in §1.4 between restriction of representations and the Jacquet functor under the geometric Casselman-Shalika equivalence.

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2. Notation

In this section, we establish notation and conventions used throughout the paper.

2.1. **Categorical conventions and base field.**

2.1.1. Throughout, we work over an algebraically closed field k of characteristic zero. We freely use the language of higher category theory and higher algebra in the sense of [Lur09], [Lur17], [GR19]. Throughout, by a (DG) category, we mean a k-linear presentable stable $(\infty, 1)$-category. We denote by $\mathrm{DGCat}_{\mathrm{cont}}$ the category of DG-categories in which the morphisms are colimit-preserving functors. Henceforth, we refer to DG-categories simply as *categories*.

2.1.2. For a category $\mathcal{C}$ equipped with a t-structure, we let $\mathcal{C}^{\leqslant 0}$ and $\mathcal{C}^{\geqslant 0}$ the subcategories of connective and coconnective objects, respectively. We denote by $\mathcal{C}^{\heartsuit} = \mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ the heart of the t-structure.

2.2. **D-modules and functoriality.**

2.2.1. Let $\mathcal{Y}$ be a prestack locally almost of finite type in the sense of [GR19]. Following [GR17], we denote by $D(\mathcal{Y})$ the (DG-)category of D-modules on $\mathcal{Y}$.

2.2.2. *Functoriality.* Let $f : \mathcal{X} \to \mathcal{Y}$ be a map of prestacks locally almost of finite type. We have a pullback functor

$$f^! : D(\mathcal{Y}) \to D(\mathcal{X}).$$

Whenever its left adjoint is defined, we denote it by $f_!$.

If f is ind-representable, we similarly have a (continuous) pushforward functor

$$f_* : D(\mathcal{X}) \to D(\mathcal{Y}).$$

Whenever its left adjoint is defined, we denote it by f^*. Recall that both $f_!$ and f^* are defined on holonomic D-modules.

2.3. **Lie theory notation.**

2.3.1. Let G be a connected reductive complex algebraic group over k. We assume throughout that G has a simply-connected derived subgroup. This is a standard assumption that simplifies the definition of Drinfeld's compactifications and Zastava spaces. To remove this hypothesis, consult [ABB⁺05, §4.1] or [Sch15, §7].

We choose a splitting $T \subseteq B$ and an opposing Borel B^- so that $B^- \cap B = T$. We let N, N^- denote the unipotent radicals of B, B^-. Denote by $\mathfrak{g}, \mathfrak{b}^-, \mathfrak{n}, \mathfrak{n}^-, \mathfrak{t}$ the corresponding Lie algebras. We let $\check{G}$ be the Langlands dual group of G.

2.3.2. We let $\check{\Lambda}, \Lambda$ (resp. $\check{\Lambda}^+, \Lambda^+$) be the lattice of characters and cocharacters of T (resp. dominant characters and dominant cocharacters). The reason for choosing this notation is that cocharacters appear more in this paper than characters, making it more convenient to reserve the 'checks' for the characters. Let W be the finite Weyl group of G.

We denote the coroots of G by Φ, and we write $\mathcal{J} = \mathcal{J}_G$ for the set of positive simple coroots of G and by $\mathcal{J}^{\mathrm{neg}}$ the set of negative simple coroots. For each $i \in \mathcal{J}$, we have a corresponding simple coroot α_i and simple root $\check{\alpha}_i$.

2.3.3. *Parabolic notation.* Let $P \subseteq G$ be a standard parabolic subgroup with unipotent radical N_P and Levi quotient M. We let $\mathfrak{p}, \mathfrak{n}_P, \mathfrak{m}$ be the corresponding Lie algebras. Write $\mathcal{J}_M \subseteq \mathcal{J}$ for the subset of the Dynkin diagram corresponding to M. We let $\check{\Lambda}_{G,P}, \Lambda_{G,P}$ be the lattices of characters and cocharacters, respectively, for the torus $M^{\mathrm{ab}} := M/[M,M]$. There is a natural map

$$\Lambda \to \Lambda_{G,P} \tag{2.3.1}$$

induced by $T \to M^{\mathrm{ab}}$. By simply-connectedness of $[G,G]$, the kernel of the above map is spanned by α_i, where $i \in \mathcal{J}_M \subseteq \mathcal{J}$ is a simple coroot corresponding to a vertex in the Dynkin diagram for M. We let $\Lambda^{\mathrm{pos}}_{G,P} \subseteq \Lambda_{G,P}$ be the monoid spanned by linear combinations with non-negative coefficients of the images of α_i for $i \in \mathcal{J} \backslash \mathcal{J}_M$ under the map (2.3.1). Write $\Lambda^{\mathrm{neg}}_{G,P} := -\Lambda^{\mathrm{pos}}_{G,P}$.

3. The main geometric players

Fix a smooth projective curve X over k.

3.1. **Affine Grassmannians and configuration spaces.** In this subsection, we review the various versions of the factorizable affine grassmannian needed in the paper. We refer to [BFGM02] for a more detailed discussion of the spaces introduced. We freely use the language of chiral/factorization theory as in [Ras15].

3.1.1. Associated to the lattice $\Lambda_{G,P}$ we have the scheme $\mathrm{Conf}_{G,P}$ parameterizing divisors on X with coefficients in $\Lambda^{\mathrm{neg}}_{G,P}$. It splits as a disjoint union

$$\mathrm{Conf}_{G,P} = \coprod_{\theta \in \Lambda^{\mathrm{neg}}_{G,P}} X^{\theta},$$

where X^{θ} parameterizes $\Lambda^{\mathrm{neg}}_{G,P}$-valued divisors on X of total degree θ. Note that $\mathrm{Conf}_{G,P}$ is naturally a monoid under addition of divisors.

3.1.2. Denote by $\mathrm{Gr}_{M,x}$ the affine Grassmannian for M at a point $x \in X$. For an M-bundle $\mathcal{P}_M$ and an M-representation W, we let $W_{\mathcal{P}_M}$ be the induced vector bundle on X. Mostly, we will consider representations of the form $W = V^{N_P}$ for a G-representation V.

3.1.3. We let $\mathrm{Gr}^+_{M,x}$ be the subscheme of $\mathrm{Gr}_{M,x}$ parameterizing $(\mathcal{P}_M, \phi) \in \mathrm{Gr}_{M,x}$, where:

- $\mathcal{P}_M$ is an M-bundle on X.
- ϕ is a trivialization away from x

$$\phi : \mathcal{P}_{M|X-x} \xrightarrow{\simeq} \mathcal{P}^0_{M|X-x}$$

such that for every G-representation V, the induced map of vector bundles

$$V^{N_P}_{\mathcal{P}_{M|X-x}} \xrightarrow{\simeq} V^{N_P}_{\mathcal{P}^0_{M|X-x}}$$

is regular on D_x.

3.1.4. The connected components of Gr_M are indexed by $\pi_1^{\mathrm{alg}}(M) := \Lambda/\mathbb{Z}\mathcal{J}_M$, the algebraic fundamental group of M (i.e., the coweight lattice of M modulo its coroot lattice). Since $[G, G]$ is simply-connected, we have $\pi_1^{\mathrm{alg}}(M) = \Lambda_{G,P}$. For $\theta \in \Lambda_{G,P}$, we write Gr^θ_M for the corresponding connected component.

Moreover, let $\mathrm{Gr}^{+,\theta}_M = \mathrm{Gr}^+_M \cap \mathrm{Gr}^\theta_M$. By construction (see e.g. [BG99, Prop. 6.2.3]), we have

$$\mathrm{Gr}^{+,\theta}_M \neq \varnothing$$

if and only if $\theta \in \Lambda^{\mathrm{neg}}_{G,P}$.

3.1.5. Denote by $\mathrm{Gr}_{M,\mathrm{Ran}}$ the Beilinson-Drinfeld affine Grassmannian living over $\mathrm{Ran} = \mathrm{Ran}_X$, the Ran space of X. We may similarly define a factorization space $\mathrm{Gr}^+_{M,\mathrm{Ran}}$ over Ran with fiber at $x \in X$ given by $\mathrm{Gr}^+_{M,x}$.

3.1.6. For an algebraic group H, we will also consider the group prestacks

$$\mathfrak{L}^+_{\mathrm{Ran}}H \text{ and } \mathfrak{L}_{\mathrm{Ran}}H$$

of *arcs* and *loops* into H over Ran, respectively. More precisely, the fiber of $\mathfrak{L}^+_{\mathrm{Ran}}H$ over a point x_I of Ran is a map from the formal completion of x_I along X to H, and the fiber of $\mathfrak{L}_{\mathrm{Ran}}H$ over a point x_I is a map from the complement of x_I in the *affinization* of the formal completion of x_I along X into H.

Note we have an equivalence

$$\mathrm{Gr}_{H,\mathrm{Ran}} \xrightarrow{\sim} \mathfrak{L}_{\mathrm{Ran}}(H)/\mathfrak{L}^+_{\mathrm{Ran}}(H)$$

as spaces over Ran. Throughout this paper, we sheafify quotients of prestack by algebraic groups in the fppf topology.

3.1.7. *Version living over configuration space.* Denote by $\mathrm{Gr}_{M,\mathrm{Conf}}$ the factorization space over $\mathrm{Conf}_{G,P}$ parameterizing triples $(D, \mathcal{P}_M, \phi)$, where $D \in \mathrm{Conf}_{G,P}$ is a $\Lambda^{\mathrm{neg}}_{G,P}$-valued divisor on X, $\mathcal{P}_M$ is an M-bundle, and ϕ is a trivialization of $\mathcal{P}_M$ away from the support of D.[13]

[13]We write $\mathrm{Gr}_{M,\mathrm{Conf}}$ instead of the more cumbersome notation $\mathrm{Gr}_{M,\mathrm{Conf}_{G,P}}$.

3.1.8. For an algebraic group H we have versions $\mathfrak{L}^+_{\mathrm{Conf}}H$ and $\mathfrak{L}_{\mathrm{Conf}}H$ of the arc and loop spaces, living over $\mathrm{Conf}_{G,P}$. These split as disjoint unions:

$$\mathfrak{L}^+_{\mathrm{Conf}}H = \coprod_{\theta\in\Lambda^{\mathrm{neg}}_{G,P}} \mathfrak{L}^+_{X^\theta}H \text{ and } \mathfrak{L}_{\mathrm{Conf}}H = \coprod_{\theta\in\Lambda^{\mathrm{neg}}_{G,P}} \mathfrak{L}_{X^\theta}H.$$

Here the index X^θ is used to denote the connected component living over the component X^θ of $\mathrm{Conf}_{G,P}$.

Concretely, $\mathfrak{L}_{\mathrm{Conf}}H$ parameterizes a divisor $D \in \mathrm{Conf}_{G,P}$ and a map from the punctured disk around the support of D to H. The arc group $\mathfrak{L}^+_{\mathrm{Conf}}H$ is defined similarly.

3.1.9. Take $H = M$. We define the group prestack $\mathfrak{L}_{\mathrm{Conf}}M^+ \subseteq \mathfrak{L}_{\mathrm{Conf}}M$ by the requirement that the diagram

$$\begin{array}{ccc} \mathfrak{L}_{\mathrm{Conf}}M^+ & \longrightarrow & \mathfrak{L}_{\mathrm{Conf}}M \\ \downarrow & & \downarrow \\ \mathrm{Gr}^+_{M,\mathrm{Conf}} & \longrightarrow & \mathrm{Gr}_{M,\mathrm{Conf}} \end{array}$$

be Cartesian. For $\theta \in \Lambda^{\mathrm{neg}}_{G,P}$, we let $\mathfrak{L}_{X^\theta}M^+$ be the pullback of $\mathfrak{L}_{\mathrm{Conf}}M^+$ to X^θ.

3.1.10. We have a natural factorization space $\mathrm{Gr}^+_{M,\mathrm{Conf}}$ over $\mathrm{Conf}_{G,P}$ with fiber at $\theta \cdot x$ given by $\mathrm{Gr}^{+,\theta}_{M,x}$.[14] For example, if $P = B$ is a Borel subgroup, then

$$(\mathrm{Gr}^+_{M,\mathrm{Conf}})^{\mathrm{red}} \simeq \mathrm{Conf}_{G,B}\,.$$

We let:

$$\mathrm{Gr}^+_{M,X^\theta} = \mathrm{Gr}^+_{M,\mathrm{Conf}} \underset{\mathrm{Conf}_{G,P}}{\times} X^\theta.$$

3.1.11. Observe that we have a natural action

$$\mathrm{Ran} \curvearrowright \mathrm{Gr}_{M,\mathrm{Ran}}$$

given by $\underline{y} \cdot (\underline{x}, \mathcal{P}_M, \phi) = (\underline{y} \cup \underline{x}, \mathcal{P}, \phi)$, where $\underline{x}, \underline{y} \in \mathrm{Ran}$. Moreover, the action preserves $\mathrm{Gr}^+_{M,\mathrm{Ran}}$. In other words, $\mathrm{Gr}_{M,\mathrm{Ran}}$ and $\mathrm{Gr}^+_{M,\mathrm{Ran}}$ are *unital* factorization spaces.

We similarly have an action:

$$\mathrm{Conf}_{G,P} \curvearrowright \mathrm{Gr}_{M,\mathrm{Conf}}\,.$$

3.1.12. *Warning.* Note that this action does not preserve $\mathrm{Gr}^+_{M,\mathrm{Conf}}$.

---

[14] There is another space which deserves to be called $\mathrm{Gr}^+_{M,\mathrm{Conf}}$; namely the factorization space over $\mathrm{Conf}_{G,P}$ with fiber at $\theta \cdot x$ given by $\mathrm{Gr}^+_{M,x}$. This space will not appear in the paper, however, and so no confusion is likely to occur.

3.1.13. We have the following basic lemma:

Lemma 3.1.13.1. *The prestacks*

$$\mathrm{Gr}_{M,\mathrm{Ran}} \,/\, \mathrm{Ran}, \quad \mathrm{Gr}_{M,\mathrm{Conf}} \,/\, \mathrm{Conf}_{G,P}$$

become isomorphic after sheafification in the fppf topology. Under this isomorphism, the composition

$$\mathrm{Gr}^+_{M,\mathrm{Conf}} \to \mathrm{Gr}_{M,\mathrm{Conf}} \,/\, \mathrm{Conf}_{G,P} \simeq \mathrm{Gr}_{M,\mathrm{Ran}} \,/\, \mathrm{Ran}$$

maps isomorphically onto $\mathrm{Gr}^+_{M,\mathrm{Ran}} \,/\, \mathrm{Ran}$.

Proof. Consider the prestack $\mathrm{Gr}_{M,\mathrm{gen}}$ whose S-points for an affine scheme S parameterize an M-bundle $\mathcal{P}_M$ on $S \times X$ together with a trivialization ϕ on *some* domain $U \subseteq S \times X$. We remind that a domain is an open subscheme such that for every k-point $s \in S$, its restriction to X is dense (equivalently, non-empty). We have natural maps:

$$\mathrm{Gr}_{M,\mathrm{Ran}} \,/\, \mathrm{Ran} \to \mathrm{Gr}_{M,\mathrm{gen}} \leftarrow \mathrm{Gr}_{M,\mathrm{Conf}} \,/\, \mathrm{Conf}_{G,P} \,.$$

We claim that these maps become isomorphism after sheafification in the fppf topology. It suffices to check this for S-points where S is an affine scheme of finite type. By [Bar12, Lemma 5.5.1], every domain $U \subseteq S \times X$ is fppf locally the complement of a union of graphs. Similarly, by Lemma 3.2.7 in *loc.cit*, every domain is Zariski locally the complement of a divisor.

To prove the second part of the lemma, define $\mathrm{Gr}^+_{M,\mathrm{gen}}$ in the obvious way. We have maps

$$\mathrm{Gr}^+_{M,\mathrm{Ran}} \,/\, \mathrm{Ran} \to \mathrm{Gr}^+_{M,\mathrm{gen}} \leftarrow \mathrm{Gr}^+_{M,\mathrm{Conf}} \,. \tag{3.1.1}$$

By construction, the left-most map is an isomorphism after fppf sheafification. Let us construct an explicit inverse to the right-most map. Thus, let $(U \subseteq S \times X, \mathcal{P}_M, \phi)$ be an S-point of $\mathrm{Gr}^+_{M,\mathrm{gen}}$. We get an associated $\Lambda^{\mathrm{pos}}_{G,P}$-valued divisor D on X. Concretely, for every character $\check{\nu}$ of M of the form $\check{\nu} = V^{N_P}$ for some G-representation V, D is the divisor such that

$$\mathcal{L}^{\check{\nu}}_{\mathcal{P}_M} \simeq \mathcal{O}_{S\times X}(-\langle D, \check{\nu}\rangle)$$

compatibly with the embedding into $\mathcal{O}_{S\times X} \simeq \mathcal{L}^{\check{\nu}}_{\mathcal{P}^0_M}$. We claim that for every G-representation V, the embedding of coherent sheaves

$$\phi_V : V^{N_P}_{\mathcal{P}_M} \hookrightarrow V^{N_P}_{\mathcal{P}^0_M}$$

is an isomorphism away from the support of D. Indeed, it suffices to check this after taking determinants of both vector bundles, where the assertion in turn follows from the definition of D.

This provides a map

$$\mathrm{Gr}^+_{M,\mathrm{gen}} \to \mathrm{Gr}^+_{M,\mathrm{Conf}},$$

which is easily seen to be the inverse of the right-most map of (3.1.1).

□

3.2. **Factorization via twisted arrows: categories.** In the next subsections, we review the construction of factorization algebras (resp. factorization categories) from a commutative algebra (resp. symmetric monoidal category). Many of the constructions follow [Gai21, §2].

3.2.1. *Twisted arrows.* Denote by fSet the category of non-empty finite sets under surjection. Note that any surjection $\phi : I \twoheadrightarrow J$ induces a closed embedding

$$\Delta_\phi : X^J \hookrightarrow X^I$$

in the natural way.

3.2.2. We let TwArr be the category whose objects are surjections

$$\phi : I \twoheadrightarrow J,$$

where $I, J \in \mathrm{fSet}$. Morphisms between $(I_1 \twoheadrightarrow J_1)$ and $(I_2 \twoheadrightarrow J_2)$ are given by commutative diagrams

$$\begin{array}{ccc} I_1 & \xrightarrow{\phi_1} & J_1 \\ {\scriptstyle \psi_I}\downarrow & & \uparrow{\scriptstyle \psi_J} \\ I_2 & \xrightarrow{\phi_2} & J_2. \end{array} \tag{3.2.1}$$

Both ψ_I and ψ_J are required to be surjective.

3.2.3. Let $\mathcal{C}$ be a unital symmetric monoidal category. We may construct a unital factorization category $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ over Ran whose fiber at $x \in X$ is $\mathcal{C}$ itself. Namely, let:

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} := \underset{(I \twoheadrightarrow J) \in \mathrm{TwArr}}{\mathrm{colim}} \mathcal{C}^{\otimes I} \otimes D(X^J), \tag{3.2.2}$$

where for any commutative diagram (3.2.1), the corresponding functors between the categories are induced by:

- $\Delta_{\psi_J,*} : D(X^{J_1}) \to D(X^{J_2})$.
- The natural map $\mathcal{C}^{\otimes I_1} \to \mathcal{C}^{\otimes I_2}$ coming from ψ_I and the monoidal structure on $\mathcal{C}$.

It is easy to see that the unital structure on $\mathcal{C}$ as a symmetric monoidal category defines a natural unital structure on $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ as a factorization category. Concretely, for $K \in \mathrm{fSet}$, the unital structure is induced by the maps

$$D(X^K) \otimes \mathcal{C}^{\otimes I} \otimes D(X^J) \to \mathcal{C}^{\otimes I \sqcup K} \otimes D(X^{J \sqcup K}),$$

where $\mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes I \sqcup K}$ is given by inserting the unit and $D(X^K) \otimes D(X^J) \to D(X^{J \sqcup K})$ is exterior product. These combine to give an action

$$D(\mathrm{Ran}) \curvearrowright \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}},$$

where $D(\mathrm{Ran})$ is considered as a symmetric monoidal category under convolution (i.e., pushforward along the map $\mathrm{Ran} \times \mathrm{Ran} \to \mathrm{Ran}, \ (\underline{x}, \underline{y}) \mapsto \underline{x} \cup \underline{y}$).

3.2.4. *Variant: Fixing I.* Let $I \in \mathrm{fSet}$. We define the category $\mathrm{TwArr}_{I/}$ whose objects are maps

$$I \twoheadrightarrow J \twoheadrightarrow K,$$

and where morphisms are commutative diagrams:

$$\begin{array}{ccccc} I & \twoheadrightarrow & J_1 & \twoheadrightarrow & K_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \uparrow \\ I & \twoheadrightarrow & J_2 & \twoheadrightarrow & K_2. \end{array}$$

It is not difficult to see that we have an isomorphism:

$$\mathrm{Fact}(\mathcal{C})_I := \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} D(X^I) \simeq \underset{(I \twoheadrightarrow J \twoheadrightarrow K) \in \mathrm{TwArr}_{I/}}{\mathrm{colim}} \mathcal{C}^{\otimes J} \otimes D(X^K). \tag{3.2.3}$$

3.2.5. *Pointwise convolution.* Let $I \twoheadrightarrow J \twoheadrightarrow K$. Observe that each term

$$\mathcal{C}^{\otimes J} \otimes D(X^K)$$

in the colimit (3.2.3) is symmetric monoidal category in $D(X^I)$-mod. Since the transition functors are moreover symmetric monoidal, we see that $\mathrm{Fact}(\mathcal{C})_I$ is equipped with a symmetric monoidal structure over X^I.

Taking the limit over $I \in \mathrm{fSet}$, we see that $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ is equipped with a symmetric monoidal structure over Ran. We refer to this monoidal structure

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \to \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$$

as *pointwise convolution.* Note in particular that the pointwise convolution of two elements in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ supported on $x, y \in X$, respectively, with $x \neq y$ is zero.

Unless stated otherwise, we will only consider $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ with its monoidal structure given by poinstwise convolution.

3.2.6. We have the following basic lemma:

Lemma 3.2.6.1. *The functor*

$$\mathcal{C} \mapsto \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$$

is symmetric monoidal. That is, we have a canonical symmetric monoidal equivalence:

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{D})_{\mathrm{Ran}} \xrightarrow{\simeq} \mathrm{Fact}(\mathcal{C} \otimes \mathcal{D})_{\mathrm{Ran}}.$$

Proof. It is easy to see that we have a canonically defined symmetric monoidal functor

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{D})_{\mathrm{Ran}} \to \mathrm{Fact}(\mathcal{C} \otimes \mathcal{D})_{\mathrm{Ran}},$$

and we need to check it is an equivalence. Since the functor is a functor of factorization categories, we may check the map is an isomorphism over $X \subseteq \mathrm{Ran}$. But

$$D(X) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \simeq \mathcal{C} \otimes D(X)$$

by (3.2.3), and similarly for $\mathcal{D}$, so this is clear. □

Remark 3.2.6.2. The above lemma also formally gives the pointwise symmetric monoidal structure on $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$.

3.3. **Factorization via twisted arrows: algebras.**

3.3.1. Let $\mathcal{A} \in \mathrm{CommAlg}^{\mathrm{un}}(\mathcal{C})$ be a unital commutative algebra object in $\mathcal{C}$. We may associate a unital factorization algebra $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ whose !-fiber at $x \in X$ is $\mathcal{A}$ itself. By !-fiber of $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ at x, we mean the functor

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \simeq D(\mathrm{Ran}_X) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \xrightarrow{i_x^! \otimes \mathrm{id}} D(\{x\}) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \simeq \mathcal{C}$$

evaluated on $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$, where $i_x : \{x\} \to \mathrm{Ran}$ is the inclusion.

Namely, we set:

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} := \underset{(I \twoheadrightarrow J) \in \mathrm{TwArr}}{\mathrm{colim}} \mathcal{A}^{\boxtimes I} \boxtimes \omega_{X^J}.$$

Here, $\mathcal{A}^{\boxtimes I} \boxtimes \omega_{X^J} \in \mathcal{C}^{\otimes I} \otimes D(X^J)$. For a commutative diagram (3.2.1), the colimit is formed with respect to the induced maps:

- $\Delta_{\psi_J,*}(\omega_{X^{J_1}}) \to \omega_{X^{J_2}}$.

- $m_{I_1 \twoheadrightarrow I_2}(\mathcal{A}^{\boxtimes I_1}) \to \mathcal{A}^{\boxtimes I_2}$ coming from the algebra structure on $\mathcal{A}$, where $m_{I_1 \twoheadrightarrow I_2}$ is the multiplication map $\mathcal{C}^{\otimes I_1} \to \mathcal{C}^{\otimes I_2}$.

The unital structure on $\mathcal{A}$ naturally defines a unital structure on $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$ as a factorization algebra.

3.3.2. It is easy to see that the functor

$$\mathcal{A} \mapsto \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$$

is right-lax symmetric monoidal. In particular, if $\mathcal{A}, \mathcal{B} \in \mathrm{CommAlg}^{\mathrm{un}}(\mathcal{C})$, we have a canonical map of commutative factorization algebras:

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}} \otimes \mathrm{Fact}^{\mathrm{alg}}(\mathcal{B})_{\mathrm{Ran}} \to \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A} \otimes \mathcal{B})_{\mathrm{Ran}} \tag{3.3.1}$$

taking place in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$.

Lemma 3.3.2.1. *The map (3.3.1) is an isomorphism. In particular, the functor*

$$\mathcal{A} \mapsto \mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})_{\mathrm{Ran}}$$

is symmetric monoidal.

Proof. We need to check that the map (3.3.1) is an isomorphism. Since (3.3.1) is a map of factorization algebras, it suffices to check that it is an isomorphism when restricted to $X \subseteq \mathrm{Ran}$. But there it follows from the fact that the restriction of $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ to X is given by

$$\mathcal{A} \boxtimes \omega_X \in \mathcal{C} \otimes D(X) \simeq D(X) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}},$$

and similarly for $\mathrm{Fact}(\mathcal{B})^{\mathrm{alg}}$. ☐

3.3.3. From the above lemma, we see that $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ is a commutative algebra object in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$. Moreover, if $\mathcal{A} \in \mathrm{CommBiAlg}(\mathcal{C}) := \mathrm{CommAlg}^{\mathrm{un}}(\mathrm{CoAlg}(\mathcal{C}))$ is a bialgebra object in $\mathcal{C}$ that is commutative as a unital algebra object, then $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ is a bialgebra object in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$.

3.3.4. *Coalgebra analogue.* If $\mathcal{A} \in \mathrm{CoCommCoAlg}^{\mathrm{un}}(\mathcal{C})$ is a unital cocommutataive coalgebra object in $\mathcal{C}$, we may similarly define a unital factorization coalgebra $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})_{\mathrm{Ran}}$ in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$ whose cofiber (or $*$-fiber) at x is $\mathcal{A}$ itself.[15] Namely:

$$\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A}) := \underset{(I \twoheadrightarrow J) \in \mathrm{TwArr}^{\mathrm{op}}}{\mathrm{colim}} \mathcal{A}^{\boxtimes I} \boxtimes \underline{k}_{X^J}.$$

Here, $\underline{k}$ denotes the constant sheaf. The maps between the terms in the colimits are dual to those of §3.3.1. Similar to above, $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})$ has a natural structure of a cocommutative coalgebra object in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$. Moreover, if $\mathcal{A}$ is a bialgebra which is cocommutative as a coalgebra, then $\mathrm{Fact}(\mathcal{A})^{\mathrm{coalg}}_{\mathrm{Ran}}$ is a cocommutative bialgebra object in $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$.

[15] Similar to before, $*$-fiber at x means the functor $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \simeq D(\mathrm{Ran}_X) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \xrightarrow{i_x^*} D(\{x\}) \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \simeq \mathcal{C}$ evaluated on $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})$. In general, this functor takes values in the pro-category $\mathrm{Pro}(\mathcal{C})$. However, when evaluated on $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})$, it is easily seen to factor through $\mathcal{C} \subseteq \mathrm{Pro}(\mathcal{C})$.

3.3.5. Finally, we note the following. Let

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran,indep}} := \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Vect}$$

be the *independent* category of $\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}$. Here, we consider $D(\mathrm{Ran})$ with its convolution monoidal structure. The action $D(\mathrm{Ran}) \rightharpoonup \mathrm{Vect}$ is given by pushforward along $p_{\mathrm{Ran}} : \mathrm{Ran} \to \mathrm{pt}$, which is symmetric monoidal.

!-pullback along p_{Ran} defines a fully faithful embedding

$$\mathrm{Fact}(\mathcal{C})_{\mathrm{Ran,indep}} \hookrightarrow \mathrm{Fact}(\mathcal{C})_{\mathrm{Ran}}.$$

It is clear that $\mathrm{Fact}^{\mathrm{alg}}(\mathcal{A})$ (resp. $\mathrm{Fact}^{\mathrm{coalg}}(\mathcal{A})$) lies in the image of this functor.

3.4. **Geometric Satake and factorization algebras associated to nilpotent radicals.**

3.4.1. Fix a parabolic subgroup $\check{P}$ of $\check{G}$ with Levi $\check{M}$. We set

$$\mathrm{Rep}(\check{M})_{\mathrm{Ran}} := \mathrm{Fact}(\mathrm{Rep}(\check{M}))_{\mathrm{Ran}}.$$

3.4.2. Denote by $\check{\mathfrak{n}}_P$ the nilradical of $\check{\mathfrak{p}} = \mathrm{Lie}(\check{P})$, and let

$$C^\bullet(\check{\mathfrak{n}}_P) \in \mathrm{Rep}(\check{M})$$

be the cohomological Chevalley complex of $\check{\mathfrak{n}}_P$ considered as a representation of $\check{M}$. Note that $C^\bullet(\check{\mathfrak{n}}_P)$ is a commutative unital algebra object in $\mathrm{Rep}(\check{M})$. Let

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}} = \mathrm{Fact}^{\mathrm{alg}}(C^\bullet(\check{\mathfrak{n}}_P))_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

be the corresponding commutative factorization algebra.

3.4.3. Dually, let

$$C_\bullet(\check{\mathfrak{n}}_P)_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})$$

be the homological Chevalley complex of $\check{\mathfrak{n}}_P$. Let

$$\Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{coalg}}(C_\bullet(\check{\mathfrak{n}}_P)) \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

be the corresponding factorization algebra. It has the structure of a cocommutative coalgebra.

3.4.4. Let

$$U(\check{\mathfrak{n}}_P) \in \mathrm{Rep}(\check{M})$$

be the universal enveloping algebra of $\check{\mathfrak{n}}_P$ and define

$$\mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{coalg}}(U(\check{\mathfrak{n}}_P)) \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}.$$

It has the structure of a cocommutative bialgebra, cf. §3.3.3.

Similarly, let

$$\mathcal{O}(\check{N}_P) \in \mathrm{Rep}(\check{M})$$

be the commutative algebra given by functions on $\check{N}_P$. We let

$$\mathcal{O}(\check{N}_P)_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}_P)) \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}.$$

It has the structure of a commutative bialgebra.

3.4.5. Let

$$\mathrm{Sph}_{M,\mathrm{Ran}}$$

be the factorization category associated to the spherical category of M. That is

$$\mathrm{Sph}_{M,\mathrm{Ran}} = \operatorname*{colim}_{I \in \mathrm{fSet}} \mathrm{Sph}_{M,I},$$

where

$$\mathrm{Sph}_{M,I} = D(\mathfrak{L}^+_{X^I} M \backslash \mathfrak{L}_{X^I} M / \mathfrak{L}^+_{X^I} M).$$

Here, $\mathfrak{L}^+_{X^I} M = \mathfrak{L}^+_{\mathrm{Ran}} M \underset{\mathrm{Ran}}{\times} X^I$ (resp. $\mathfrak{L}_{X^I} M$) parameterizes an I-tuple $x_I \in X^I$ and a map $D_{x_I} \to M$ (resp. $\overset{\circ}{D}_{x_I} \to M$).

3.4.6. We have a full subcategory

$$\mathrm{Sph}^+_{M,\mathrm{Ran}} \subseteq \mathrm{Sph}_{M,\mathrm{Ran}}$$

consisting of D-modules supported on $\mathrm{Gr}^+_{M,\mathrm{Ran}}$. Note that the (pointwise) monoidal structure on $\mathrm{Sph}_{M,\mathrm{Ran}}$ restricts to one on $\mathrm{Sph}^+_{M,\mathrm{Ran}}$.

3.4.7. We also have versions living over the configuration space:

$$\mathrm{Sph}^+_{M,\mathrm{Conf}} \subseteq \mathrm{Sph}_{M,\mathrm{Conf.}}$$

Here, the !-fiber of $\mathrm{Sph}^+_{M,\mathrm{Conf}}$ at some $\theta \cdot x \in \mathrm{Conf}_{G,P}$ is

$$D(\mathfrak{L}^+_x M \backslash \mathrm{Gr}^{+,\theta}_{M,x}).$$

Let us give a concrete description of $\mathrm{Sph}^+_{M,\mathrm{Conf}}$, which also makes it evident that the category is equipped with a natural external convolution monoidal structure.

3.4.8. Recall the group prestack $\mathfrak{L}_{\mathrm{Conf}} M^+$ over $\mathrm{Conf}_{G,P}$ from §3.1.9. Observe that we have:

$$\mathfrak{L}_{\mathrm{Conf}} M^+ / \mathfrak{L}^+_{\mathrm{Conf}} M = \mathrm{Gr}^+_{M,\mathrm{Conf}}.$$

Moreover, we have:

$$\mathrm{Sph}^+_{M,\mathrm{Conf}} = D(\mathfrak{L}^+_{\mathrm{Conf}} M \backslash \mathfrak{L}_{\mathrm{Conf}} M^+ / \mathfrak{L}^+_{\mathrm{Conf}} M).$$

Note that we have a natural decomposition

$$\mathrm{Sph}^+_{M,\mathrm{Conf}} = \bigoplus_{\theta \in \Lambda^{\mathrm{neg}}_{G,P}} \mathrm{Sph}^+_{M,X^\theta},$$

where $\mathrm{Sph}^+_{M,X^\theta} := \mathrm{Sph}^+_{M,\mathrm{Conf}} \underset{D(\mathrm{Conf}_{G,P})}{\otimes} D(X^\theta)$.

3.4.9. There is a natural action (not relative to $\mathrm{Conf}_{G,P}$):

$$\mathfrak{L}_{\mathrm{Conf}} M^+ \times \mathrm{Gr}^+_{M,\mathrm{Conf}} \to \mathrm{Gr}^+_{M,\mathrm{Conf}}$$

changing the trivialization at the punctured disk. Note that the underlying divisors of $\mathrm{Conf}_{G,P}$ get added under this action.

3.4.10. The external convolution structure on $\mathrm{Sph}^+_{M,\mathrm{Conf}}$ is defined by !-pull, $*$-push along the correspondence:

$$\begin{array}{ccc} \mathfrak{L}^+_{\mathrm{Conf}}M\backslash\mathfrak{L}_{\mathrm{Conf}}M^+ \overset{\mathfrak{L}^+_{\mathrm{Conf}}M}{\times} \mathrm{Gr}^+_{M,\mathrm{Conf}} & \longrightarrow & \mathfrak{L}^+_{\mathrm{Conf}}M\backslash\mathfrak{L}_{\mathrm{Conf}}M^+/\mathfrak{L}^+_{\mathrm{Conf}}M \\ \downarrow & & \\ \mathfrak{L}^+_{\mathrm{Conf}}M\backslash\mathfrak{L}_{\mathrm{Conf}}M^+/\mathfrak{L}^+_{\mathrm{Conf}}M \times \mathfrak{L}^+_{\mathrm{Conf}}M\backslash\mathfrak{L}_{\mathrm{Conf}}M^+/\mathfrak{L}^+_{\mathrm{Conf}}M. & & \end{array}$$

We note that the usual argument shows that the horizontal map of the above correspondence is stratified semismall. In particular, convolution is t-exact for the natural perverse t-structure on $\mathrm{Sph}^+_{M,\mathrm{Conf}}$ given in §3.4.17.

3.4.11. *The factorizable naive geometric Satake functor.* By [Ras21, §6], we have a monoidal functor

$$\mathrm{Sat}^{\mathrm{nv}}_{\mathrm{Ran}} : \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{Sph}_{M,\mathrm{Ran}}$$

that restricts to the usual (naive) geometric Satake functor $\mathrm{Sat}^{\mathrm{nv}}$ on fibers and that is t-exact when restricted to X^I. We will denote by $\mathrm{Sat}^{\mathrm{nv}}_{X^I}$ the restriction of $\mathrm{Sat}^{\mathrm{nv}}_{\mathrm{Ran}}$ to $D(X^I)$.

By abuse of notation, we also denote by

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$$

the corresponding factorization algebras from §3.4 under $\mathrm{Sat}^{\mathrm{nv}}_{\mathrm{Ran}}$.

Lemma 3.4.11.1. *The factorization algebras*

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Ran}},\ \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$$

are supported on

$$\mathrm{Sph}^+_{M,\mathrm{Ran}} \subseteq \mathrm{Sph}_{M,\mathrm{Ran}.}$$

Proof. Recall that we may also regard $\Lambda_{G,P}$ as the character lattice of $Z(\check{M})^\circ$, the connected component of the identity of the center of $\check{M}$. Here, $\Lambda^{\mathrm{pos}}_{G,P}$ becomes the positive span of the simple positive roots not contained in $\mathfrak{m}$. We will prove the assertion for $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$, the proof for the remaining factorization algebras is similar.

Let w_0^M be the longest element in the Weyl group of M. By [BG99, Prop. 6.2.3], it suffices to check that if ν is an $\check{M}$-dominant root occurring in $\check{\mathfrak{n}}_P$, then $w_0^M(\nu)$ is a sum of positive roots in $\check{G}$. However, this simply follows from the fact that $\check{M}$ stabilizes $\check{\mathfrak{n}}_P$; in particular, $w_0^M(\nu)$ will be a sum of positive roots in $\check{\mathfrak{n}}_P$. □

3.4.12. Let

$$\mathrm{Sph}^+_{M,\mathrm{Ran},\mathrm{indep}} := \mathrm{Sph}^+_{M,\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Vect},$$

as in §3.3.5. By Lemma 3.1.13.1, we have an equivalence:

$$\mathrm{Sph}^+_{M,\mathrm{Ran},\mathrm{indep}} \simeq \mathrm{Sph}^+_{M,\mathrm{Conf}.} \tag{3.4.1}$$

By unitality of the factorization algebras, we may view $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}, \Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}, \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}, \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ as factorization algebras in $\mathrm{Sph}^+_{M,\mathrm{Conf}}$. To avoid confusion, we write

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}, \Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}, \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}, \mathcal{O}(\check{N}_P)_{\mathrm{Conf}}$$

for the corresponding factorization algebras in $\mathrm{Sph}^+_{M,\mathrm{Conf}}$.

3.4.13. By construction, the !-fiber of $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ and $\mathcal{O}(\check{N}_P)_{\mathrm{Conf}}$ at $\theta\cdot x \in \mathrm{Conf}_{G,P}$ is

$$C^\bullet(\check{\mathfrak{n}}_P)^\theta, \mathcal{O}(\check{N}_P)^\theta \in \mathrm{Sph}^+_{M,x};$$

that is, the image under geometric Satake of the θ-graded piece of $C^\bullet(\check{\mathfrak{n}}_P)$ and $\mathcal{O}(\check{N}_P)$, respectively. We have analogous assertions for $\Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}, \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Cof}}$ with respect to $*$-fibers.

We let

$$\Omega(\check{\mathfrak{n}}_P)_{X^\theta}, \Upsilon(\check{\mathfrak{n}}_P)_{X^\theta}, \mathfrak{U}(\check{\mathfrak{n}}_P)_{X^\theta}, \mathcal{O}(\check{N}_P)_{X^\theta} \in \mathrm{Sph}^+_{M,X^\theta}$$

be the respective restrictions of the factorization algebras to $X^\theta \subseteq \mathrm{Conf}_{G,P}$.

3.4.14. Note that the pointwise convolution structure on $\mathrm{Sph}_{M,\mathrm{Ran}}$ restricts to one on $\mathrm{Sph}^+_{M,\mathrm{Ran}}$.

Lemma 3.4.14.1. *Under the equivalence (3.4.1), pointwise convolution structure goes to external convolution.*

Proof. The pointwise convolution structure on $\mathrm{Sph}^+_{M,\mathrm{Ran}}$ is given by pull-push along the correspondence

$$\begin{array}{ccc}
\mathfrak{L}^+_{\mathrm{Ran}}M\backslash\mathfrak{L}_{\mathrm{Ran}}M^+ \overset{\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathrm{Gr}^+_{M,\mathrm{Ran}} & \longrightarrow & \mathfrak{L}^+_{\mathrm{Ran}}M\backslash\mathfrak{L}_{\mathrm{Ran}}M^+/\mathfrak{L}^+_{\mathrm{Ran}}M \\
\downarrow & & \\
\mathfrak{L}^+_{\mathrm{Ran}}M\backslash\mathfrak{L}_{\mathrm{Ran}}M^+/\mathfrak{L}^+_{\mathrm{Ran}}M \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash\mathfrak{L}_{\mathrm{Ran}}M^+/\mathfrak{L}^+_{\mathrm{Ran}}M. & &
\end{array}$$

Note that the above is a correspondence of unital factorization spaces over Ran. Taking the quotient by Ran, we obtain the convolution diagram for $\mathrm{Sph}^+_{M,\mathrm{Ran},\mathrm{indep}}$. However, by Lemma 3.1.13.1, this is precisely the convolution diagram for $\mathrm{Sph}^+_{M,\mathrm{Conf}}$. □

3.4.15. By the above lemma, $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ is equipped with a commutative algebra structure in $\mathrm{Sph}^+_{M,\mathrm{Conf}}$ with its external convolution structure. Similar assertions hold for $\Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ and $\mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$.

3.4.16. By construction, $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ is a locally compact object of $\mathrm{Sph}^+_{M,\mathrm{Conf}}$, and we have

$$\mathbb{D}(\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}) \simeq \Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}. \tag{3.4.2}$$

Similarly, we have:

$$\mathbb{D}(\mathcal{O}(\check{N}_P)_{\mathrm{Conf}}) \simeq \mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}. \tag{3.4.3}$$

3.4.17. *t-structure.* We define a t-structre on $\mathrm{Sph}^+_{M,\mathrm{Conf}}$ by the requirement that the forgetful functor

$$\mathrm{Sph}^+_{M,\mathrm{Conf}} \to D(\mathrm{Gr}^+_{M,\mathrm{Conf}}) \tag{3.4.4}$$

is right t-exact. The argument in [Ber21, Lemma 2.1.15] shows that (3.4.4) is t-exact.

Lemma 3.4.17.1. *The factorization algebras $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ and $\Upsilon(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ are perverse.*

Proof. It suffices to show that $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ is perverse by (3.4.2).

Let $F^0\check{\mathfrak{n}}_P = \check{\mathfrak{n}}_P, F^{-1}\check{\mathfrak{n}}_P = [\check{\mathfrak{n}}_P,\check{\mathfrak{n}}_P], F^{-2}\check{\mathfrak{n}}_P = [F^{-1}\check{\mathfrak{n}}_P,\check{\mathfrak{n}}_P]$ etc. Let $\check{\mathfrak{n}}^i_P = F^{-i}\check{\mathfrak{n}}_P/F^{-i-1}\check{\mathfrak{n}}_P$. Denoting by $\star$ the convolution structure on $\mathrm{Sph}^+_{M,\mathrm{Conf}}$. Then $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ carries a filtration with associated graded

$$\underset{i}{\star}\Omega(\check{\mathfrak{n}}^i_P)_{\mathrm{Conf}}.$$

Since convolution is t-exact, we may assume that $\check{\mathfrak{n}}_P$ is abelian. Decompose $\check{\mathfrak{n}}_P = \bigoplus_{\alpha\in\Lambda^{\mathrm{pos}}_{G,P}} \check{\mathfrak{n}}_{P,\alpha}$ into its weight spaces for the action of $Z(\check{M})$. Then we need to show that each

$$\Omega(\check{\mathfrak{n}}_{P,\alpha})_{\mathrm{Conf}}$$

is perverse, where we consider $\check{\mathfrak{n}}_{P,\alpha}$ as an abelian Lie algebra. Recall that $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\theta}$ denotes the restriction of $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{\mathrm{Conf}}$ to $X^\theta \subseteq \mathrm{Conf}_{G,P}$. Then $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\theta}$ is non-zero only if $\theta = n\cdot\alpha$, in which case

$$\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\theta} = (\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\alpha})^{\star n}.$$

Thus, it suffices to show that $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\alpha}$ is perverse. By construction, it is supported over the main diagonal $X \hookrightarrow X^\alpha$, $x \mapsto \alpha\cdot x$. Moreover, it is easy to see that $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\alpha}$ is ULA over X. Thus, we simply need to check that the !-fibers of $\Omega(\check{\mathfrak{n}}_{P,\alpha})_{X^\alpha}$ at a point $\alpha\cdot x$ is concentrated in perverse degree 1. But this fiber is the image of $C^\bullet(\check{\mathfrak{n}}_{P,\alpha})^\alpha = \check{\mathfrak{n}}_{P,\alpha}[-1]$ under geometric Satake. ☐

3.4.18. We record the following lemmas for later use:

Lemma 3.4.18.1. *For $\theta \neq 0$, the sheaf $\mathfrak{U}(\check{\mathfrak{n}}_P)_{X^\theta}$ is concentrated in perverse degrees $\geqslant 1$.*

Proof. Since $\mathfrak{U}(\check{\mathfrak{n}}_P)_{\mathrm{Conf}}$ is a factorization algebra, it suffices to show that its $*$-pullback along the diagonal

$$X \to X^\theta, \ \ x \mapsto \theta\cdot x$$

is concentrated in perverse degrees $\geqslant 1$. Denote by $\mathfrak{U}(\check{\mathfrak{n}}_P)_{\Delta^\theta}$ the resulting sheaf over X. As in the proof of Lemma 3.4.17.1, the sheaf $\mathfrak{U}(\check{\mathfrak{n}}_P)_{\Delta^\theta}$ is ULA over X. Moreover, its $*$-fiber at some x is given by the image of

$$U(\check{\mathfrak{n}}_P)^\theta \in \mathrm{Rep}(\check{M})^\heartsuit$$

under geometric Satake. Since the latter is perverse, this forces $\mathfrak{U}(\check{\mathfrak{n}}_P)_{\Delta^\theta}$ to live in strictly positive cohomological perverse degrees. ☐

Lemma 3.4.18.2. *For $\theta \neq 0$, the sheaf $\mathcal{O}(\check{\mathfrak{n}}_P)_{X^\theta}$ is concentrated in perverse degrees $\leqslant -1$.*

Proof. This follows from a similar argument to that of Lemma (3.4.18.1), or alternatively by applying *loc.cit* and Verdier duality. ☐

3.5. **Drinfeld's compactification.**

3.5.1. Associated to our smooth projective curve X, we have the moduli stack Bun_G of principal G-bundles on X. Additionally, we have the moduli stacks Bun_P and Bun_M of principal P and M-bundles, respectively. The quotient map $P \to M$ and the inclusion $P \hookrightarrow G$ induce morphisms:

$$q : \mathrm{Bun}_P \to \mathrm{Bun}_M;$$

$$p : \mathrm{Bun}_P \to \mathrm{Bun}_G\,.$$

The stack Bun_M has connected components Bun_M^η indexed by $\eta \in \pi_1^{\mathrm{alg}}(M) = \Lambda_{G,P}$. We denote by Bun_P^η the preimage of Bun_M^η in Bun_P.

3.5.2. The map $p : \mathrm{Bun}_P \to \mathrm{Bun}_G$ has a relative compactification

$$\widetilde{p} : \widetilde{\mathrm{Bun}}_P \longrightarrow \mathrm{Bun}_G,$$

namely Drinfeld's compactification, see [BG99]. The stack $\widetilde{\mathrm{Bun}}_P$ is also equipped with a map

$$\widetilde{q} : \widetilde{\mathrm{Bun}}_P \longrightarrow \mathrm{Bun}_M$$

and splits as a disjoint union

$$\widetilde{\mathrm{Bun}}_P = \coprod_{\eta \in \Lambda_{G,P}} \widetilde{\mathrm{Bun}}_P^{\eta}$$

of connected components.

3.5.3. *Stratification.* For $\theta \in \Lambda_{G,P}^{\mathrm{neg}}$, we let $\mathcal{H}_{M,\mathrm{Conf}}$ be the Hecke stack over $\mathrm{Conf}_{G,P}$ parameterizing $(D, \mathcal{P}_M^1, \mathcal{P}_M^2, \phi)$, where:

- $D \in \mathrm{Conf}_{G,P}$.
- Each $\mathcal{P}_M^i$ is an M-bundle on X.
- ϕ is an isomorphism:
$$(\mathcal{P}_M^1)_{|X-D} \simeq (\mathcal{P}_M^2)_{|X-D}.$$

We may similarly define the Hecke stack $\mathcal{H}_{\mathrm{Conf}}^+ \subseteq \mathcal{H}_{\mathrm{Conf}}$, where we require that the isomorphism ϕ extends to a regular embedding whenever twisting by V^{N_P} for every $V \in \mathrm{Rep}(G)$, analogous to the definition given in §3.1.2.

We have decompositions:

$$\mathcal{H}_{M,\mathrm{Conf}} = \coprod_{\theta \in \Lambda_{G,P}^{\mathrm{neg}}} \mathcal{H}_{M,X^\theta}, \quad \mathcal{H}_{M,\mathrm{Conf}}^+ = \coprod_{\theta \in \Lambda_{G,P}^{\mathrm{neg}}} \mathcal{H}_{M,X^\theta}^+.$$

3.5.4. We have a natural map

$$\mathrm{act} : \mathcal{H}_M^+ \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P \to \widetilde{\mathrm{Bun}}_P \tag{3.5.1}$$

by modifying the M-bundle at D. For $\theta \in \Lambda_{G,P}^{\mathrm{neg}}$, we let ι_θ be the composition:

$$\mathcal{H}_{M,X^\theta}^+ \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \to \mathcal{H}_{M,\mathrm{Conf}}^+ \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_P \to \widetilde{\mathrm{Bun}}_P.$$

Proposition 3.5.4.1 ([BFGM02], Prop. 1.9)**.** *Each map ι_θ is a locally closed embedding and these stratify $\widetilde{\mathrm{Bun}}_P$.*

3.5.5. Henceforth, we write

$${}_\theta\widetilde{\mathrm{Bun}}_P := \mathcal{H}_{M,X^\theta}^+ \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P.$$

In particular, ${}_0\widetilde{\mathrm{Bun}}_P$ identifies with Bun_P itself, and in this case we write $j = \iota_0$ for the open embedding:

$$j : \mathrm{Bun}_P \hookrightarrow \widetilde{\mathrm{Bun}}_P.$$

We will also consider a finer stratification over each connected component of $\widetilde{\mathrm{Bun}}_P$. Namely, $\widetilde{\mathrm{Bun}}_P^\eta$ is stratified by the substacks

$${}_\theta\widetilde{\mathrm{Bun}}_P^\eta \simeq \mathcal{H}_{M,X^\theta}^+ \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P^{\theta+\eta},$$

whose embedding into $\widetilde{\mathrm{Bun}}_P^\eta$ we also denote by ι_θ.

4. Parabolic semi-infinite IC-sheaf

In this section, we introduce the factorizable parabolic semi-infinite IC-sheaf. We define it in terms of Drinfeld-Plücker and Hecke structures, and so we start this section by discussing the latter notions.

4.1. **Pointwise Hecke- and Drinfeld-Plücker structures.** In this subsection, we give an overview of (parabolic) Hecke and Drinfeld-Plücker structures. We follow closely that of [Gai21, §5.4] where the case of a principal parabolic is considered.

4.1.1. For a scheme X, we write $\mathcal{O}(X)$ for the derived global sections of the structure sheaf of X.

4.1.2. *Hecke structures.* Denote by $\mathrm{Res}^{\check{G}}_{\check{M}}$ the restriction functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M}).$$

Consider the regular representation

$$\mathcal{O}(\check{G}) \in \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}) \tag{4.1.1}$$

considered as a $\check{G} \times \check{G}$-representation. By slight abuse of notation, we continue to $\mathcal{O}(\check{G})$ for the image of (4.1.1) under the functor

$$\mathrm{Res}^{\check{G}}_{\check{M}} \otimes \mathrm{id} : \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}).$$

4.1.3. Let $\mathcal{C}$ be a $(\mathrm{Rep}(\check{M}), \mathrm{Rep}(\check{G}))$-bimodule category. Since $\mathrm{Rep}(\check{G})$ is symmetric monoidal, a right $\mathrm{Rep}(\check{G})$-action gives a left action. As such, we consider $\mathcal{C}$ as acted on by $\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G})$ on the left.

Consider the category

$$\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) := \mathcal{O}(\check{G})\text{-mod}(\mathcal{C})$$

of modules for $\mathcal{O}(\check{G})$ in $\mathcal{C}$. For $c \in \mathcal{C}$, we refer to a lift of c to an object of $\mathcal{O}(\check{G})\text{-mod}(\mathcal{C})$ as a *Hecke structure* on c.

4.1.4. *Drinfeld-Plücker structures.* Let $\check{N}_P$ denote the unipotent radical of $\check{P}$. Consider the functor

$$(-)^{\check{N}_P} : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M}), \quad V \mapsto V^{\check{N}_P}$$

of taking invariants against $\check{N}_P$. Here we take *non-derived* invariants.[16]

Let

$$\overline{\check{N}_P\backslash\check{G}} := \mathrm{Spec}(H^0(\mathcal{O}(\check{N}_P\backslash\check{G})))$$

be the (parabolic) basic affine space. We have:

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \simeq \bigoplus_{\lambda} (V^{\lambda})^{\check{N}_P} \otimes (V^{\lambda})^*$$

as $\check{M} \times \check{G}$-representations, where the sum is over all dominant coweights of G. We remind that $(V^{\lambda})^{\check{N}_P}$ identifies with the irreducible representation $\check{M}$ of highest weight λ.

[16] So for example the trivial representation maps to the trivial representation.

4.1.5. Consider the category

$$\mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) := \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})\text{-mod}(\mathcal{C}).$$

Concretely, for $c \in \mathcal{C}$, the datum of a lift of c to $\mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C})$ consists of a family of maps

$$V^{\check{N}_P} \star c \to c \star V, \quad V \in \mathrm{Rep}(\check{G})$$

satisfying Drinfeld-Plücker identities as in [Gai21, §5.3]. We refer to such a lift as a *Drinfeld-Plücker* structure on c.

4.1.6. The map

$$\check{G} \to \overline{\check{N}_P\backslash\check{G}}$$

induces a homomorphism

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \to \mathcal{O}(\check{G})$$

and hence a forgetful functor

$$\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}). \tag{4.1.2}$$

The functor (4.1.2) admits a left adjoint:

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}), \quad c \mapsto \mathcal{O}(\check{G}) \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})}{\otimes} c.$$

4.2. **Factorizable Hecke- and Drinfeld-Plücker structures.**

4.2.1. From the algebras

$$\mathcal{O}(\check{G}), \mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \in \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}),$$

we get the algebras (cf. §3.3.1):

$$\mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}))_{\mathrm{Ran}}, \mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{G}))_{\mathrm{Ran}} \in (\mathrm{Rep}(\check{M})\otimes\mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \simeq \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

We use the shorthand notation:

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}))_{\mathrm{Ran}}, \quad \mathcal{O}(\check{G})_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{G}))_{\mathrm{Ran}}.$$

Moreover, recall the coalgebra

$$\mathcal{O}(\check{N}_P)_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}},$$

cf. §3.4.4.

4.2.2. Suppose that $\mathcal{C}$ is a category over Ran equipped with a $(\mathrm{Rep}(\check{M})_{\mathrm{Ran}}, \mathrm{Rep}(\check{G})_{\mathrm{Ran}})$-bimodule structure. As in §4.1.3, we consider it as a category acted on by $\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ on the left.

We write

$$\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) := \mathcal{O}(\check{G})_{\mathrm{Ran}}\text{-mod}(\mathcal{C}), \quad \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) := \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

As in §4.1, we have a map of algebras

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \to \mathcal{O}(\check{G})_{\mathrm{Ran}},$$

inducing a forgetful functor

$$\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}). \tag{4.2.1}$$

The functor (4.2.1) admits a left adjoint:

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}), \;\; c \mapsto \mathcal{O}(\check{G})_{\mathrm{Ran}} \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}}{\otimes} c. \tag{4.2.2}$$

4.2.3. Recall that $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ has a natural structure of a coalgebra object in $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$. The action

$$\check{N}_P \curvearrowright \check{G}$$

induces a coaction of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ on $\mathcal{O}(\check{G})_{\mathrm{Ran}}$. It follows that we have a coaction of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ on the monad

$$\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} \circ \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}).$$

Corollary 4.2.3.1. *Let* $c \in \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C})$*. Then* $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ *coacts on* $\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} \circ \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(c)$*.*

4.2.4. *Enhanced Drinfeld-Plücker structures.* Next, we introduce an intermediate structure that sits between a Hecke strcture and a Drinfeld-Plücker structure.

From the algebra

$$\mathcal{O}(\check{N}_P\backslash\check{G}) \in \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}),$$

we get the algebra

$$\mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}} := \mathrm{Fact}^{\mathrm{alg}}(\mathcal{O}(\check{N}_P\backslash\check{G}))_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

Define

$$\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) := \mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

We refer to objects in $\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C})$ as objects in $\mathcal{C}$ equipped with an *enhanced Drinfeld-Plücker* structure. The map $\check{N}_P\backslash\check{G} \to \overline{\check{N}_P\backslash\check{G}}$ induces a forgetful functor

$$\mathrm{Oblv}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}),$$

which admits a left adjoint:

$$\mathrm{Ind}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}} : \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}).$$

Similarly, the map $\check{G} \to \check{N}_P\backslash\check{G}$ induces a forgetful functor

$$\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}} : \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C})$$

with a left adjoint:

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}} \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}).$$

By construction, the composition

$$\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}} \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Oblv}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}} \mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C})$$

recovers the functor $\mathrm{Oblv}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}$, and similarly for the left adjoints.

Note that because $\check{N}_P\backslash\check{G} \to \overline{\check{N}_P\backslash\check{G}}$ is an open embedding and the target is affine, it follows that the forgetful functor $\mathrm{Oblv}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}$ is fully faithful. That is, an upgrade of a Drinfeld-Plücker structure to an enhanced Drinfeld-Plücker structure is a property.

4.2.5. We now describe more explicitly what it means to have a Hecke structure and an enhanced Drinfeld-Plücker structure.

Tautologically, we have identifications

$$\begin{aligned}\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) &\simeq \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}}{\otimes} \mathcal{C} \\ &\simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}}{\otimes} \mathcal{C},\end{aligned}$$

where in the last term we consider $\mathcal{C}$ as acted on by $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ via the monoidal functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Res}^{\check{G}}_{\check{M}} \otimes \mathrm{id}} \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

This shows that for $c \in \mathcal{C}$, the datum of a lift to $\mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C})$ is equivalent to the datum of an identification between the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on c and the action induced by the restriction functor $\mathrm{Res}^{\check{G}}_{\check{M}}$. That is, we require a family of isomorphisms

$$\mathrm{Res}^{\check{G}}_{\check{M}}(V) \star c \simeq c \star V, \quad V \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

satisfying natural compatibilities (see e.g. [Gai21, §5]).

4.2.6. For enhanced Drinfeld-Plücker structures, we have an equivalence:

$$\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \simeq \mathrm{Rep}(\check{P})_{\mathrm{Ran}} \underset{\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}}{\otimes} \mathcal{C}. \tag{4.2.3}$$

Moreover, consider the functor

$$C^\bullet(\check{\mathfrak{n}}_P, -) : \mathrm{Rep}(\check{P})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

induced by restricting along $\check{P} \to \check{G}$ and applying (derived) Lie algebra cohomology against $\check{\mathfrak{n}}_P$.[17]

This functor induces a symmetric monoidal equivalence

$$\mathrm{Rep}(\check{P})_{\mathrm{Ran}} \xrightarrow{\simeq} \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{Rep}(\check{M})_{\mathrm{Ran}}),$$

where $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$ is as in §3.4.2. In particular, we have an action:

$$\mathrm{Rep}(\check{P})_{\mathrm{Ran}} \curvearrowright \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

From (4.2.3), we obtain an equivalence

$$\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}}{\otimes} \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C}), \tag{4.2.4}$$

where we consider $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C})$ as acted on by $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ via the monoidal functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Res}^{\check{G}}_{\check{P}} \otimes \mathrm{id}} \mathrm{Rep}(\check{P})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

From the map $\check{N}_P\backslash\check{G} \to \check{N}_P\backslash \mathrm{pt}$, we obtain a forgetful functor

$$\mathrm{Oblv} : \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

[17] We remark that even though taking invariants against $\check{\mathfrak{n}}_P$ is not symmetric monoidal, it is lax symmetric monoidal, and so still gives rise to a functor on the level of twisted arrows.

We remind that $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$ is an algebra object of $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$, and the latter acts on $\mathcal{C}$ via the symmetric monoidal functor $\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, inserting the unit on the second factor.

In particular, the existence of an enhanced Drinfeld-Plücker structure implies the existence of a module structure for $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$. The equivalence (4.2.4) shows that for $c \in \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C})$, the datum of a lift to $\mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C})$ is equivalent to the datum of an identification between the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on c and the action induced by the monoidal functor $C^\bullet(\check{\mathfrak{n}}_P, -) : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{Rep}(\check{M})_{\mathrm{Ran}})$. That is, we require a family of isomorphisms

$$C^\bullet(\check{\mathfrak{n}}_P, V) \underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star} c \simeq c \star V, \quad V \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

satisfying natural higher compatibilities.

4.2.7. We highlight one further compatibility between the constructions. The functor

$$\mathrm{Triv}_{\mathcal{O}(\check{N}_P)} : \mathcal{C} \to \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C})$$

endowing an object $c \in \mathcal{C}$ with the trivial comodule structure admits a right adjoint:

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)} : \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C}) \to \mathcal{C}$$

given by taking invariants against $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$. The objects in the image of this functor naturally admit a module structure for $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$, and as such, we get an enhanced functor:

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P),\mathrm{enh}} : \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C}) \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

Lemma 4.2.7.1. *Let $\mathcal{C}$ be a category acted on by $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$. Suppose $\mathcal{C}$ is dualizable as a module category for $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$. Then the functor*

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P),\mathrm{enh}} : \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C}) \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C})$$

is an equivalence.

Proof.

Step 1. The assertion is clear when $\mathcal{C} = \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ by the usual Koszul duality between $\mathcal{O}(\check{N}_P)$ and $\Omega(\check{\mathfrak{n}}_P)$. Thus, it remains to prove that the canonical functors:

$$\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathrm{Rep}(\check{M})_{\mathrm{Ran}}) \underset{\mathrm{Rep}(\check{M})_{\mathrm{Ran}}}{\otimes} \mathcal{C} \to \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C});$$

$$\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathrm{Rep}(\check{M})_{\mathrm{Ran}}) \underset{\mathrm{Rep}(\check{M})_{\mathrm{Ran}}}{\otimes} \mathcal{C} \to \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C})$$

are equivalences.

Step 2. Consider the following setup. Let $\mathcal{A}$ be a symmetric monoidal category and $\mathcal{C}$ a module category for $\mathcal{A}$. Let $A \in \mathcal{A}$ be a cocommutative coalgebra object, and let $B \in \mathcal{A}$ be a commutative algebra object. We may consider the functors:

$$A\text{-comod}(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{C} \to A\text{-comod}(\mathcal{C});$$

$$B\text{-mod}(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{C} \to B\text{-mod}(\mathcal{C}).$$

The second functor is always an equivalence, see [GR19, Corollary 8.5.7]. We claim that the first functor is an equivalence under the assumption that $\mathcal{C}$ is dualizable as an $\mathcal{A}$-module category. Indeed, we need to check that the forgetful functor

$$A\text{-comod}(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{C} \to \mathcal{C}$$

is comonadic. In turn it suffices to check that it is conservative. But this holds whenever $\mathcal{C}$ is dualizable as an $\mathcal{A}$-module category. □

4.2.8. By construction, the composition

$$\mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Ind}_{\mathrm{DrPl}_{\check{M},\check{G}}}^{\mathrm{Hecke}_{\check{M},\check{G}}}} \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\text{Cor. 4.2.3.1}} \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}\text{-comod}(\mathcal{C}) \xrightarrow{\mathrm{Inv}_{\mathcal{O}(\check{N}_P),\mathrm{enh}}} \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C})$$

coincides with the functor

$$\mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Ind}_{\mathrm{DrPl}_{\check{M},\check{G}}}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}} \mathrm{EnhDrPl}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Oblv}} \Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}\text{-mod}(\mathcal{C}).$$

4.3. **Semi-infinite category.**

4.3.1. We now specialize to the case of interest, namely the semi-infinite category defined below. Before doing so, we need a suitable modification of the geometric Satake equivalence for M.

For θ an element of $\in \Lambda_{G,P}$, which we remind identifies with the character lattice of the connected component of the center of $\check{M}$, denote by $e^\theta \in \mathrm{Rep}(Z(\check{M})^\circ)$ the corresponding character of $Z(\check{M})^\circ$. Let $2\rho_M$ denote the sum of positive coroots of M.

The category $\mathrm{Rep}(\check{M})$ decomposes as a direct sum

$$\mathrm{Rep}(\check{M}) \simeq \bigoplus_{\theta\in\Lambda_{G,P}} \mathrm{Rep}(\check{M})^\theta,$$

where $\mathrm{Rep}(\check{M})^\theta$ denotes the category of $\check{M}$-representations in which $Z(\check{M})^{\mathrm{circ}}$ acts via θ.

4.3.2. Denote by $\mathcal{S}^\theta$ the functor:

$$\mathcal{S}^\theta : \mathrm{Rep}(\check{M})^\theta \to \mathrm{Rep}(\check{M})^\theta, \;\; V \mapsto V[-\langle 2(\rho_G - \rho_M), \theta\rangle]. \tag{4.3.1}$$

These functors combine to a symmetric monoidal autoequivalence:

$$\mathcal{S} = \underset{\theta\in\Lambda_{G,P}}{\oplus} \mathcal{S}^\theta : \mathrm{Rep}(\check{M}) \to \mathrm{Rep}(\check{M}).$$

This functor makes sense factorizably as well. Namely, we have an evident symmetric monoidal factorizable functor

$$\mathcal{S}_{\mathrm{Ran}} : \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

that recovers $\mathcal{S}$ on fibers. In terms of the presentation (3.2.2), $\mathcal{S}_{\mathrm{Ran}}$ is defined by taking the colimit of the functors

$$\mathrm{Rep}(\check{M})^{\otimes I} \otimes D(X^J) \xrightarrow{\mathcal{S}^{\otimes I}\otimes \mathrm{id}} \mathrm{Rep}(\check{M})^{\otimes I} \otimes D(X^J) \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}}.$$

4.3.3. Let $\mathfrak{L}N_P$ denote the loop group of N_P and consider its factorizable version $\mathfrak{L}_{\mathrm{Ran}}N_P \to \mathrm{Ran}$. We remind that it is a (factorizable) ind-group scheme. We define:

$$\mathrm{SI}_{P,\mathrm{Ran}} := D(\mathfrak{L}_{\mathrm{Ran}}N_P\mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$

We refer to the above category as the *semi-infinite category* (associated to P). Note that it is a full subcategory of $D(\mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$.

4.3.4. The action

$$\mathrm{Sph}_{M,\mathrm{Ran}} \curvearrowright D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}})$$

induces an action

$$\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \xrightarrow{\mathcal{S}_{\mathrm{Ran}}} \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{Sph}_{M,\mathrm{Ran}} \curvearrowright \mathrm{SI}_{P,\mathrm{Ran}}\,. \tag{4.3.2}$$

Here, the second functor is the (naive) geometric Satake functor. Henceforth, when we talk about the action of $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ on $D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}})$ or $\mathrm{SI}_{P,\mathrm{Ran}}$, we always mean (4.3.2).

4.3.5. We have a right action

$$D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}) \curvearrowleft \mathrm{Sph}_{G,\mathrm{Ran}} \leftarrow \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

As such, the categories

$$D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}), \;\; \mathrm{SI}_{P,\mathrm{Ran}}$$

are equipped with a $(\mathrm{Rep}(\check{M})_{\mathrm{Ran}}, \mathrm{Rep}(\check{G})_{\mathrm{Ran}})$-bimodule structure and hence fit the framework of §4.2.

4.3.6. The unit section

$$\mathrm{Ran} \to \mathrm{Gr}_{G,\mathrm{Ran}}$$

induces a section

$$s_{\mathrm{Ran}} : \mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}} M \to \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}.}$$

Let

$$\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}} := (s_{\mathrm{Ran}})_!(\omega_{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}} M}) \in D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$

4.3.7. In Appendix B, we construct a Drinfeld-Plücker structure on $\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$. As such, we may define:

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} := \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}) \in \mathrm{Hecke}_{\check{M},\check{G}}(D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}})).$$

We refer to the above sheaf as the (factorizable) semi-infinite sheaf associated to P.

4.3.8. By Corollary 4.2.3.1, we have a coaction:

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \to \mathcal{O}(\check{N}_P)_{\mathrm{Ran}} \star \mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}\,.$$

4.4. **Stratification.** The purpose of this section is to calculate the restriction of the semi-infinite IC-sheaf to a natural stratification.

4.4.1. Let $S^0_{P,\mathrm{Ran}} \subseteq \mathrm{Gr}_{G,\mathrm{Ran}}$ be the $\mathfrak{L}_{\mathrm{Ran}} N_P$-orbit along the unit section $\mathrm{Ran} \to \mathrm{Gr}_{G,\mathrm{Ran}}$. That is, $S^0_{P,\mathrm{Ran}}$ is the factorizable ind-scheme defined as follows: for an affine test scheme T, a map $T \to \mathrm{Gr}_{G,\mathrm{Ran}}$ corresponding to a triple $(\underline{x}, \mathcal{P}_G, \phi)$ factors through $S^0_{P,\mathrm{Ran}}$ if and only if for every G-representation V, the meromorphic map of vector bundles (regular on $T \times X \backslash \Gamma$, where Γ is the union of graphs of the maps comprising $\underline{x}$)

$$V^{N_P}_{\mathcal{P}^0_G} \to V_{\mathcal{P}^0_G} \to V_{\mathcal{P}_G} \tag{4.4.1}$$

extends to an injective map of vector bundles over $T \times X$ (that is, the meromorphic map extends to a regular map of coherent sheaves over $T \times X$ with cokernel flat over $T \times X$).

4.4.2. Let $\widetilde{S}^0_{P,\mathrm{Ran}} \subseteq \mathrm{Gr}_{G,\mathrm{Ran}}$ be the 'closure' of $S^0_{P,\mathrm{Ran}}$ in $\mathrm{Gr}_{G,\mathrm{Ran}}$. That is, a map $T \to \mathrm{Gr}_{G,\mathrm{Ran}}$ factors through $\widetilde{S}^0_{P,\mathrm{Ran}}$ if and only if the meromorphic map (4.4.1) extends to an injective map of coherent sheaves (with cokernel flat over T).

4.4.3. The prestack $\widetilde{S}^0_{P,\mathrm{Ran}}$ admits a stratification indexed by $\Lambda^{\mathrm{neg}}_{G,P}$, by bounding the zeroes of (4.4.1). More precisely, for $\theta \in \Lambda^{\mathrm{neg}}_{G,P}$, let $S^\theta_{P,\mathrm{Ran}}$ be the prestack defined as follows: for an affine test scheme T, a map $T \to \widetilde{S}^0_{P,\mathrm{Ran}}$ corresponding to a triple $(\underline{x}, \mathcal{P}_G, \phi)$ factors through $S^\theta_{P,\mathrm{Ran}}$ if and only if there exists a colored divisor $D \in X^\theta(T)$ such that for every $\lambda \in \Lambda^{\mathrm{neg}}_{G,P}$, the map (4.4.1) extends to an injective map of vector bundles:

$$V^{N_P}_{\mathcal{P}^0_G}(\lambda(D)) \to V_{\mathcal{P}_G}.$$

Note that each stratum $S^\theta_{P,\mathrm{Ran}}$ is $\mathfrak{L}_{\mathrm{Ran}} N_P$-stable.

4.4.4. We denote by j^θ the corresponding locally closed embedding:

$$j^\theta : S^\theta_{P,\mathrm{Ran}} \hookrightarrow \widetilde{S}^0_{P,\mathrm{Ran}}.$$

By taking the zeroes of (4.4.1), we obtain a map

$$S^\theta_{P,\mathrm{Ran}} \to X^\theta. \tag{4.4.2}$$

4.4.5. For $\theta \in \Lambda^{\mathrm{neg}}_{G,P}$, let

$$(X^\theta \times \mathrm{Ran})^{\subseteq} \subseteq X^\theta \times \mathrm{Ran}$$

be the subprestack whose T-points parameterize pairs $(D, \underline{x})$ of a colored divisor on $T \times X$ of total degree θ together with a map $\underline{x} : T \to \mathrm{Ran}$ such that D is set-theoretically supported on the union of the graphs of $\underline{x}$.

4.4.6. Recall the factorization space $\mathrm{Gr}^+_{M,\mathrm{Ran}}$ defined in §3.1.5. Recall similarly the space $\mathrm{Gr}^+_{M,\mathrm{Conf}} \to \mathrm{Conf}_{G,P}$ defined in §3.1.7. By Lemma 3.1.13.1, we have a natural map $\mathrm{Gr}^+_{M,\mathrm{Ran}} \to \mathrm{Gr}^+_{M,\mathrm{Conf}}$.

We let $\mathrm{Gr}^+_{M,X^\theta}$ and $\mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}}$ denote the respective pullbacks of $\mathrm{Gr}^+_{M,\mathrm{Conf}}$ and $\mathrm{Gr}^+_{M,\mathrm{Ran}}$ along $X^\theta \to \mathrm{Conf}_{G,P}$. Note that we have a natural projection map:

$$\mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}} \to (X^\theta \times \mathrm{Ran})^{\subseteq}.$$

4.4.7. By construction, the map (4.4.2) factors through a map:

$$'p^\theta : S^\theta_{P,\mathrm{Ran}} \to \mathfrak{L}_{\mathrm{Ran}} N_P \backslash S^\theta_{P,\mathrm{Ran}} \to \mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}}.$$

Moreover:

Lemma 4.4.7.1. *The functor*

$$'p^{\theta,!} : D(\mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}}) \to D(\mathfrak{L}_{\mathrm{Ran}} N_P \backslash S^\theta_{P,\mathrm{Ran}})$$

is an equivalence.

Proof. It is easy to see that the map $\mathfrak{L}_{\mathrm{Ran}} N_P \backslash S^\theta_{P,\mathrm{Ran}} \to \mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}}$ realizes the source as a unipotent gerbe over the target. ☐

4.4.8. Note that the action of $\mathfrak{L}^+_{\mathrm{Ran}} M$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ stabilizes $\widetilde{S}^0_{P,\mathrm{Ran}}$. Define

$$\mathrm{SI}^{\leqslant 0}_{P,\mathrm{Ran}} := D(\mathfrak{L}_{\mathrm{Ran}} N_P \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \widetilde{S}^0_{P,\mathrm{Ran}})$$

to be the full subcategory consisting of $\mathrm{SI}_{M,\mathrm{Ran}}$ of D-modules supported on $\widetilde{S}^0_{P,\mathrm{Ran}}$. The following lemma is proved in Appendix A:

Lemma 4.4.8.1. *The sheaf* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ *defines an object in* $\mathrm{SI}^{\leqslant 0}_{P,\mathrm{Ran}}$. *That is,* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ *is* $\mathfrak{L}_{\mathrm{Ran}} N_P$-*equivariant and is supported on* $\widetilde{S}^0_{P,\mathrm{Ran}}$.

4.4.9. Next, we describe the restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to the stratum $S^{\theta}_{\mathrm{Ran}}$.

Note that the map $'p^{\theta}$ is $\mathfrak{L}^{+}_{\mathrm{Ran}}M$-equivariant. We similarly denote by $'p^{\theta}$ the induced map:

$$'p^{\theta} : \mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash S^{\theta}_{P,\mathrm{Ran}} \to \mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash \mathrm{Gr}^{+}_{M,(X^{\theta}\times\mathrm{Ran})^{\subseteq}}.$$

We let p^{θ} denote the composition:

$$p^{\theta} : \mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash S^{\theta}_{P,\mathrm{Ran}} \xrightarrow{'p^{\theta}} \mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash \mathrm{Gr}^{+}_{M,(X^{\theta}\times\mathrm{Ran})^{\subseteq}} \to \mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash \mathrm{Gr}^{+}_{M,X^{\theta}}.$$

The following proposition is proved in Section 4.5 below.

Proposition 4.4.9.1. *We have a canonical isomorphism:*

$$j^{\theta,*}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq p^{\theta,!}(\mathcal{O}(\check{N}_P)_{X^{\theta}})[-\langle 2(\rho_G-\rho_M),\theta\rangle].$$

4.5. **Stratification computation.** In this section, we prove Proposition 4.4.9.1.

4.5.1. Denote by $j=j^0$ the open embedding:

$$S^{0}_{P,\mathrm{Ran}} \hookrightarrow \widetilde{S}^{0}_{P,\mathrm{Ran}}.$$

For convenience, write:

$$\mathbf{j}_! := j_!(\omega_{S^{0}_{P,\mathrm{Ran}}}) \in D(\widetilde{S}^{0}_{P,\mathrm{Ran}}).$$

Note that $\mathbf{j}_!$ naturally defines an object of $\mathrm{SI}^{\leqslant 0}_{P,\mathrm{Ran}}$.

4.5.2. Consider the delta sheaf $\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}} \in D(\mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}})$. From its Drinfeld-Plücker structure, we may define the object:

$$\mathrm{Ind}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_G,\mathrm{Ran}}) \in \mathrm{EnhDrPl}_{\check{M},\check{G}}(D(\mathfrak{L}^{+}_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}})).$$

By §4.2.7, we have an isomorphism

$$\mathrm{Ind}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_G,\mathrm{Ran}}) \simeq \mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$$

of $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$-modules. By Lemma 4.4.8.1, the sheaf $\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$ defines an object of the category $\mathrm{SI}^{\leqslant 0}_{P,\mathrm{Ran}}$.

4.5.3. Consider the diagram:

$$\begin{array}{ccc} S^{\theta}_{P,\mathrm{Ran}} & \xrightarrow{\quad j^{\theta}\quad} & \widetilde{S}^{0}_{P,\mathrm{Ran}} \\ \Big\downarrow{\scriptstyle 'p^{\theta}} & & \\ \mathrm{Gr}^{+}_{M,(X^{\theta}\times\mathrm{Ran})^{\subseteq}}. & & \end{array}$$

Let $\mathrm{pres}^{\theta} :=' p^{\theta}_! \circ j^{\theta,*}[\langle 2(\rho_G-\rho_M),\theta\rangle] : D(\widetilde{S}^{0}_{P,\mathrm{Ran}}) \to D(\mathrm{Gr}^{+}_{M,(X^{\theta}\times\mathrm{Ran})^{\subseteq}})$ denote the corresponding parabolic restriction functor. Let $\mathrm{pres} := \bigoplus_{\theta} \mathrm{pres}^{\theta}$.

We denote by $\mathrm{pres}^{\theta,x} : D(\widetilde{S}^{0}_{P,x}) \to D(\mathrm{Gr}^{+,\theta}_{M,x})$ the corresponding functor where we replace Ran by a point $x \in X$. Similarly, let $\mathrm{pres}^{x} := \bigoplus_{\theta} \mathrm{pres}^{\theta,x}$.

4.5.4. We will prove:

Proposition 4.5.4.1. *We have a canonical identification:*

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq \mathbf{j}_!.$$

In particular, the sheaf $\mathbf{j}_!$ *is equipped with a canonical module structure for* $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$ *and carries a canonical enhanced Drinfeld-Plücker structure.*

Proof. Since both sheaves in question are objects of $\mathrm{SI}^{\leqslant 0}_{P,\mathrm{Ran}}$, by Lemma 4.4.8.1, it suffices to show that

$$\mathrm{pres}^{\theta}(\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})) = 0 \tag{4.5.1}$$

for all $\theta \neq 0$ and that

$$\mathrm{pres}^{0}(\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})) \simeq \delta_{\mathrm{Gr}_{M,\mathrm{Ran}}}. \tag{4.5.2}$$

Here $\delta_{\mathrm{Gr}_{M,\mathrm{Ran}}}$ is the image of the dualizing sheaf under the unit section $\mathrm{Ran} \to \mathrm{Gr}_{M,\mathrm{Ran}}$. Note that $\mathrm{pres}^{0}(\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}))$ is a unital sheaf; that is, defines an object of $D(\mathrm{Gr}^{+,0}_{M,\mathrm{Ran}} / \mathrm{Ran}) \simeq \mathrm{Vect}$. Moreover, $\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$ is a factorization algebra. As such, we may check the identities (4.5.1) and (4.5.2) after taking the !-fiber at every $x \in X \subseteq \mathrm{Ran}$.[18]

We write $\mathcal{O}(\check{N}_P\backslash\check{G})_x$ (resp. $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x$) for the !-restriction of $\mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}}$ (resp. $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}$) to a point x, i.e., as objects of $\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G})$.

By §4.2.7, we have the identity:

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq \mathrm{Ind}^{\mathrm{EnhDrPl}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}) = \mathcal{O}(\check{N}_P\backslash\check{G})_{\mathrm{Ran}} \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}}{\otimes} \delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}.$$

As such, it suffices to establish the identities:

$$\mathrm{pres}^{\theta,x}(\mathcal{O}(\check{N}_P\backslash\check{G})_x \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x}{\otimes} \delta_{\mathrm{Gr}_{G,x}}) = 0; \tag{4.5.3}$$

$$\mathrm{pres}^{0,x}(\mathcal{O}(\check{N}_P\backslash G)_x \underset{\mathcal{O}(\overline{\check{N}_P\backslash G})_x}{\otimes} \delta_{\mathrm{Gr}_{G,x}}) = \delta_{\mathrm{Gr}_{M,x}}. \tag{4.5.4}$$

Step 1. Write the tensor product in question as a Bar complex:

$$\mathcal{O}(\check{N}_P\backslash\check{G})_x \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x}{\otimes} \delta_{\mathrm{Gr}_{G,x}} = \underset{n}{\mathrm{colim}}\, \mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{G,x}}.$$

Note that $\mathrm{pres}^{x}(\mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{G,x}})$ is by definition the image of $\mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n}$ under the functor:

$$\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}) \to \mathrm{Sph}_{M,x} \otimes \mathrm{Sph}_{G,x} \xrightarrow{\mathrm{id} \otimes \mathrm{pres}^x} \mathrm{Sph}_{M,x} \otimes \mathrm{Sph}_{M,x} \xrightarrow{-\star-} \mathrm{Sph}_{M,x}. \tag{4.5.5}$$

Here, the first functor is the geometric Satake functor, the third is convolution, and we have abused notation by also denoting $\mathrm{pres}^x : \mathrm{Sph}_{G,x} \to \mathrm{Sph}_{M,x}$ the (shifted) direct sum over all θ of the functor

[18] We remark that the functor pres commutes with taking !-fibers by hyperbolic localization: the functor pres coincides with the analogous parabolic restriction functor for the opposite parabolic P^-, replacing $*$-pull, !-push with !-pull, $*$-push, and the latter functor clearly commutes with !-fibers by base change.

of $*$-pull and !-push along:

$$\begin{array}{ccc} \mathfrak{L}_x^+ M\backslash S_x^\theta & \longrightarrow & \mathfrak{L}_x^+ G\backslash \mathfrak{L}_x G/\mathfrak{L}_x^+ G \\ \downarrow & & \\ \mathfrak{L}_x^+ M\backslash \mathfrak{L}_x M/\mathfrak{L}_x^+ M. & & \end{array}$$

Recall that by construction of the geometric Satake functor $\mathrm{Sat}^{\mathrm{nv}} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}_{G,x}$, the composition

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}^{\mathrm{nv}}} \mathrm{Sph}_{G,x} \xrightarrow{\mathrm{pres}^x} \mathrm{Sph}_{M,x}$$

coincides with the functor

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Res}^{\check{G}}_{\check{M}}} \mathrm{Rep}(\check{M}) \xrightarrow{\bigoplus_\theta \mathrm{Sat}_M^{\mathrm{nv},\theta}[-\langle 2(\rho_G-\rho_M),\theta\rangle]} \mathrm{Sph}_{M,x}, \tag{4.5.6}$$

see e.g. [MV07, §3] or [BG01, Thm. 2.2 (3)]. As such, we get:

$$\mathrm{pres}^x(\mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{G,x}}) \simeq \mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{M,x}},$$

where we now consider $\mathrm{Sph}_{M,x}$ as module category for $\mathrm{Rep}(\check{M})\otimes\mathrm{Rep}(\check{G})$ via the restriction functor $\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{id}\otimes\mathrm{Res}^{\check{G}}_{\check{M}}} \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{M})$ and the usual Satake action of the latter on $\mathrm{Sph}_{M,x}$.[19] We conclude that:

$$\mathrm{pres}^x(\mathcal{O}(\check{N}_P\backslash\check{G})_x \underset{\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x}{\otimes} \delta_{\mathrm{Gr}_{G,x}}) \simeq \operatorname*{colim}_n \mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{M,x}}. \tag{4.5.7}$$

Step 2. Consider $\delta_{\mathrm{Gr}_{M,x}}$ as a module for $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x \in \mathrm{Rep}(\check{M})\otimes\mathrm{Rep}(\check{G})$ via the augmentation. That is, the action of $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x$ on $\delta_{\mathrm{Gr}_{M,x}}$ is given by $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x \to k$ induced by the point $1 \in \overline{\check{N}_P\backslash\check{G}}$, and where we consider $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x$ as an $\check{M}$-representation via the diagonal map $\check{M} \to \check{M} \times \check{G}$.

This endows $\delta_{\mathrm{Gr}_{M,x}}$ with a Drinfeld-Plücker structure at x. This tautologically comes from an enhanced Drinfeld-Plücker structure (which we remind is a property, not a structure, cf. §4.2.4). That is:

$$\delta_{\mathrm{Gr}_{M,x}} \simeq \mathcal{O}(\check{N}_P\backslash\check{G})_x \underset{\mathcal{O}(\check{N}_P\backslash\check{G})_x}{\otimes} \delta_{\mathrm{Gr}_{M,x}} = \operatorname*{colim}_n \mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x^{\otimes n} \otimes \delta_{\mathrm{Gr}_{M,x}}. \tag{4.5.8}$$

If we can show that the transition maps in the colimits (4.5.7) and (4.5.8) agree in a coherent manner, then we are done.

In the transition maps in (4.5.7), we consider maps of two types:

- The maps $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x\otimes\delta_{\mathrm{Gr}_{M,x}} \to \delta_{\mathrm{Gr}_{M,x}}$ induced by applying pres^x to the map $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x\otimes \delta_{\mathrm{Gr}_{G,x}} \to \delta_{\mathrm{Gr}_{G},x}$ in $D(\mathfrak{L}_x^+ M\backslash \mathrm{Gr}_{G,x})$ induced by the Drinfeld-Plücker structure on $\delta_{\mathrm{Gr}_{G,x}}$. This evidently coincides with the corresponding map in the colimit (4.5.8). Moreover, since both $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x \otimes \delta_{\mathrm{Gr}_{M,x}}$ and $\delta_{\mathrm{Gr}_{M,x}}$ are in the heart of the perverse t-structure on $\mathrm{Sph}_{M,x}$, this automatically provides higher coherence.
- The maps $\mathcal{O}(\check{N}_P\backslash\check{G})_x \otimes \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x \to \mathcal{O}(\check{N}_P\backslash\check{G})_x$. By construction, these are obtained by applying the functor (4.5.5) to the natural action $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_x$ on $\mathcal{O}(\check{N}_P\backslash\check{G})_x$. These coincide with the similar maps appearing in the colimit (4.5.8).

[19] We remark that the shift by $-\langle 2(\rho_G - \rho_M, \theta)\rangle$ in the functor (4.5.6) exactly cancels the shift appearing in §4.3.2 for the action of $\mathrm{Rep}(\check{M})$ on $D(\mathfrak{L}_x^+ M\backslash \mathrm{Gr}_{G,x})$.

□

4.5.5. We may now prove Proposition 4.4.9.1:

Proof of Proposition 4.4.9.1. By $\mathfrak{L}_{\mathrm{Ran}}N_P$-equivariance of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$, we need to show that $\mathrm{pres}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$. Indeed, by unitality, the !-restriction of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ along $\mathrm{Gr}^+_{M,(X^\theta\times\mathrm{Ran})^{\subseteq}} \to \mathrm{Gr}^+_{M,\mathrm{Ran}}$ descends to $\mathrm{Gr}^+_{M,X^\theta}$.

Since pres is $\mathrm{Sph}_{M,\mathrm{Ran}}$-linear, we obtain a coaction of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$ on $\mathrm{pres}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$. Moreover, by (the proof of) Proposition 4.5.4.1, we have an isomorphism

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{pres}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})) \simeq \delta_{\mathrm{Gr}_{M,\mathrm{Ran}}}$$

as $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$-modules, where the action of $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$ on $\delta_{\mathrm{Gr}_{M,\mathrm{Ran}}}$ is trivial. The proof now follows by Koszul duality: denote by $\delta_{\mathrm{Gr}_{M,\mathrm{Ran}}} \underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star} -$ the functor of taking coinvariants for a $\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}$-module. That is, the inverse functor to the equivalence of Lemma 4.2.7.1.[20] Then:

$$\mathrm{pres}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq \delta_{\mathrm{Gr}_{M,\mathrm{Ran}}} \underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star} \mathrm{Inv}_{\mathcal{O}(\check{N}_P)}(\mathrm{pres}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})) \simeq \delta_{\mathrm{Gr}_{M,\mathrm{Ran}}} \underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star} \delta_{\mathrm{Gr}_{M,\mathrm{Ran}}} \simeq \mathcal{O}(\check{N}_P)_{\mathrm{Ran}}.$$

□

4.5.6. Finally, let us record a lemma that provides a lower bound on the perverse cohomological degrees of the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to strata. This will be used in the local-to-global comparison in Section 5.3.

By $\mathfrak{L}_{\mathrm{Ran}}N_P$-equivariance and unitality of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$, the sheaf $j^{\theta,!}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$ descends along p^θ. That is, we have

$$j^{\theta,!}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \simeq p^{\theta,!}(\mathcal{F}^\theta)$$

for some $\mathcal{F}^\theta \in D(\mathrm{Gr}^+_{M,X^\theta})$.

4.5.7. The following lemma is proved in Appendix A.2:

Lemma 4.5.7.1. *For $\theta \neq 0$, the sheaf $\mathcal{F}^\theta$ lies in perverse cohomological degrees $\geqslant 1+\langle 2(\rho_G-\rho_M),\theta\rangle$.*

Remark 4.5.7.2. It follows from Corollary 5.3.8.2 below that in fact:

$$\mathcal{F}^\theta \simeq \mathfrak{U}(\check{\mathfrak{n}}_P)_{X^\theta}[-\langle 2(\rho_G-\rho_M),\theta\rangle].$$

[20]We apply Lemma 4.2.7.1 to the category $\mathcal{C} = \mathrm{Sph}_{M,\mathrm{Ran}}$, which is easily seen to be dualizable as a module category for $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$.

5. Local-to-global comparisons

In this section, we will describe a local version of the geometric Eisenstein series functors and compare them with their global counterparts.

5.1. **The relative semi-infinite space.** Define a relative version of $\widetilde{S}^0_{P,\mathrm{Ran}}$ over Bun_M as the fiber product:

$$(\mathrm{Bun}_M \times \mathrm{Ran}) \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash \widetilde{S}^0_{P,\mathrm{Ran}} =: \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M}.$$

Here $\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M$ denotes the factorizable prestack parameterizing a point of Ran together with an M-bundle on its formal neighborhood along X. Note every such bundle is trivial fppf locally on the base. Unwinding the definition of $\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M}$, we see by Beauville-Laszlo gluing that there is a canonical map:

$$\pi_P : \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \longrightarrow \widetilde{\mathrm{Bun}}_P.$$

5.1.1. Note that $\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M}$ is stratified by the spaces

$$(\mathrm{Bun}_M \times \mathrm{Ran}) \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash S^{\theta}_{P,\mathrm{Ran}}$$

in a way compatible with the stratification on $\widetilde{\mathrm{Bun}}_P$, cf. Section 3.5.

Let $\mathrm{pr_0}$ and $\mathrm{pr_M}$ denote the projections from $\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M}$ to $\mathfrak{L}^+_{\mathrm{Ran}}M\backslash \widetilde{S}^0_{P,\mathrm{Ran}}$ and Bun_M, respectively.

5.2. **Homological contractibility results.** In this section, we will show that the map π_P is universally homologically contractible.

5.2.1. Let $\mathcal{Y}$ be a prestack equipped with a map $\mathcal{Y} \to \mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M$. Define

$${}_{\mathcal{Y}}\mathrm{Gr}_{G,\mathrm{Ran}} := \mathcal{Y} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}}$$

to be the corresponding twisted version of the Beilinson-Drinfeld affine Grassmannian. To simplify notation, we write:

$${}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} :=_{\mathrm{Bun}_M \times \mathrm{Ran}} \mathrm{Gr}_{G,\mathrm{Ran}}.$$

5.2.2. Note that ${}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}}$ parameterizes a point x_I of Ran, a G-bundle $\mathcal{P}_G$ on X, an M-bundle $\mathcal{P}_M$ on X and an identification of $\mathcal{P}_G$ with $\mathcal{P}_M \overset{M}{\times} G$ away from x_I.

Note that there is an action

$$\mathrm{Sph}_{M,\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Sph}_{G,\mathrm{Ran}} \curvearrowright D({}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}}),$$

and therefore an action

$$\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \curvearrowright D({}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}})$$

by restriction along the naive geometric Satake functor associated to the reductive group $M \times G$.

5.2.3. Define $\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}} \to \mathrm{Ran}$ to be the moduli stack parameterizing a point x_I of Ran, a G-bundle $\mathcal{P}_G$, an M-bundle $\mathcal{P}_M$, and for every G-representation V, injective meromorphic maps

$$V^{N_P}_{\mathcal{P}_M} \to V_{\mathcal{P}_G} \tag{5.2.1}$$

of coherent sheaves satisfying the Plücker relations, and which are non-degenerate away from x_I.

Inside $\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$ there is the subfunctor $\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}}$ consisting of points as above with the property that each map (5.2.1) is regular. In this case, the zeroes of the maps (5.2.1) are automatically supported on the x_I. We have a natural map:

$$\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \to \widetilde{\mathrm{Bun}}_P \times \mathrm{Ran}\,.$$

5.2.4. Note that there is a canonical map

$$\pi_{P,\mathrm{pol}} : {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \longrightarrow \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$$

taking a generic reduction of a G-bundle to an M-bundle to the resulting (meromorphic) Plücker data. The fiber of $\pi_{P,\mathrm{pol}}$ over $\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}}$ is precisely $\widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M}$. We let $\pi_{P,\mathrm{zer}}$ denote the resulting map:

$$\pi_{P,\mathrm{zer}} : \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \pi_{P,\mathrm{zer}}.$$

5.2.5. For $\theta \in \Lambda^{\mathrm{neg}}_{G,P}$, define:

$${}_{\theta}\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} := \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \underset{\widetilde{\mathrm{Bun}}_P}{\times} {}_{\theta}\widetilde{\mathrm{Bun}}_P.$$

5.2.6. The main property of $\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$ is that we have a Hecke action:

$$\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \curvearrowright D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}).$$

The morphism $\pi_{P,\mathrm{pol}}$ clearly commutes with Hecke correspondences for M and G, and hence the functor $\pi^!_{P,\mathrm{pol}}$ is equivariant for the action of $\mathrm{Rep}(\check{M})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

5.2.7. For an algebraic group H, define $\mathrm{Bun}^{\mathrm{gen}}_H$ to be the stack parameterizing a generically defined H bundle on X, see [Bar12, §2]. For any homomorphism $H \to K$ of algebraic groups, there is an evident map

$$\mathrm{Ind}^{\mathrm{gen}}_{H\to K} : \mathrm{Bun}^{\mathrm{gen}}_H \longrightarrow \mathrm{Bun}^{\mathrm{gen}}_K$$

given by induction of torsors. In particular, we obtain a map $\mathrm{Ind}^{\mathrm{gen}}_{M\to P} : \mathrm{Bun}^{\mathrm{gen}}_M \to \mathrm{Bun}^{\mathrm{gen}}_P$.

5.2.8. Before continuing, let us briefly recall the notion of universal homological contractibility from [Gai11]. A morphism $\mathcal{Y} \to \mathcal{X}$ is said to be *universally homologically contractible* if for any affine scheme S mapping to $\mathcal{X}$, the !-pullback functor

$$D(S) \longrightarrow D(S \underset{\mathcal{X}}{\times} \mathcal{Y})$$

along the projection $S \underset{\mathcal{X}}{\times} \mathcal{Y} \to S$ is fully faithful. Using that any prestack may be written as a colimit of affine schemes, we obtain that for *any* prestack $\mathcal{X}_0$ mapping to $\mathcal{X}$, we have that !-pullback

$$D(\mathcal{X}_0) \longrightarrow D(\mathcal{X}_0 \underset{\mathcal{X}}{\times} \mathcal{Y})$$

along the projection is fully faithful.

Lemma 5.2.8.1. *The induction map* $\mathrm{Ind}^{\mathrm{gen}}_{M\to P} : \mathrm{Bun}^{\mathrm{gen}}_M \to \mathrm{Bun}^{\mathrm{gen}}_P$ *is universally homologically contractible.*

Proof. Fix an S-point $\mathcal{P}_P$ of $\mathrm{Bun}_P^{\mathrm{gen}}$. We wish to show that pullback along the projection
$$S \underset{\mathrm{Bun}_P^{\mathrm{gen}}}{\times} \mathrm{Bun}_M^{\mathrm{gen}} \to S$$
is fully faithful. To this end, if
$$p : \mathscr{Q} \to U \subseteq X \times S$$
is a morphism from a prestack $\mathscr{Q}$ to a domain U in $X \times S$, we can define the prestack
$$\underline{\mathrm{GSect}}_S(X \times S, \mathscr{Q})$$
of generically defined sections of p (see [Bar12] for a definition).[21]

For a generically defined S-family of P-torsors $\mathcal{P}_P$, we can consider the quotient
$$\mathcal{P}_P \overset{P}{\times} P/M := (\mathcal{P}_P \times P/M)/P$$
where P acts diagonally. Note $\mathcal{P}_P \overset{P}{\times} P/M$ is equipped with the structure of an fppf locally trivial fibration over U with a universally homologically contractible fiber[22] given by $P/M \simeq N_P$.

Now it is easy to see that there is an isomorphism
$$\underline{\mathrm{GSect}}_S(X \times S, \mathcal{P}_P \overset{P}{\times} P/M) \xrightarrow{\sim} S \underset{\mathrm{Bun}_P^{\mathrm{gen}}}{\times} \mathrm{Bun}_M^{\mathrm{gen}}$$
commuting with the projections to S. The result now follows from [Gai11, Lemma 3.1.2] using the fact that $P/M \simeq N_P$ is an affine space (see also [Bar12, Remark 6.2.12]). □

Proposition 5.2.8.2. *The map* $\pi_{P,\mathrm{pol}} : {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \to \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$ *is universally homologically contractible.*

Proof. Forgetting the point of Ran, we obtain morphisms
$${}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \longrightarrow \mathrm{Bun}_M^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{\mathrm{gen}}}{\times} \mathrm{Bun}_G, \quad \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}} \longrightarrow \mathrm{Bun}_P^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{\mathrm{gen}}}{\times} \mathrm{Bun}_G$$
such that the diagram
$$\begin{array}{ccc} {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} & \longrightarrow & \mathrm{Bun}_M^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{\mathrm{gen}}}{\times} \mathrm{Bun}_G \\ \Big\downarrow{\scriptstyle \pi_{P,\mathrm{pol}}} & & \Big\downarrow \\ \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}} & \longrightarrow & \mathrm{Bun}_P^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{\mathrm{gen}}}{\times} \mathrm{Bun}_G \end{array}$$
is Cartesian. Here the right vertical arrow is obtained via base change from $\mathrm{Ind}_{M\to P}^{\mathrm{gen}}$. Now the result follows from Lemma 5.2.8.1. □

Corollary 5.2.8.3. *The map* $\pi_P : \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} \to \widetilde{\mathrm{Bun}}_P$ *is universally homologically contractible.*

[21] Although in [Bar12], the author only defines rational sections for $\mathscr{Q}$ living over the whole curve, the space still makes sense in this broader generality since, for example, the intersection of any two domains is a domain.

[22] Of course, in general there is no action of N_P on $\mathcal{P}_P \overset{P}{\times} P/M$ unless $\mathcal{P}_P$ is trivial, and hence $\mathcal{P}_P \overset{P}{\times} P/M$ is *not* a torsor for N_P.

Proof. We have a Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} & \longrightarrow & {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \\ {\scriptstyle \pi_{P,\mathrm{zer}}}\downarrow & & \downarrow{\scriptstyle \pi_{P,\mathrm{pol}}} \\ \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} & \longrightarrow & \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}. \end{array}$$

It is easy to see that the forgetful map

$$\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \to \widetilde{\mathrm{Bun}}_P$$

that forgets the point of Ran is *pseudo-proper*, i.e. its fiber over an affine scheme S can be written as a colimit of schemes proper over S with closed embeddings as transition maps.

Hence by [Gai21, Lemma A.2.5], it suffices to show that its fibers over field-valued points are homologically contractible. This follows, for example, from [Gai21, Prop. A.2.7]. □

5.3. **Local-to-global compatibility of IC sheaves.** In this section we verify that $\pi_P^!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$ is equivalent to the relative semi-infinite IC sheaf, up to suitable shifts.

5.3.1. There is a natural map

$$j_{P,\mathrm{pol}} : \mathrm{Bun}_P \times \mathrm{Ran} \longrightarrow \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$$

over Ran taking a P-bundle with a point x_I of Ran to the associated non-degenerate Plücker data for the induced G-bundle (see e.g. [BG99, §4.1]).

5.3.2. For any $\theta \in \Lambda_{G,P}^{\mathrm{neg}}$, define

$$\mathrm{Gr}_{P,\mathrm{Bun}_M}^{\theta} := \mathrm{Bun}_M \times \mathrm{Ran} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M \backslash S^{\theta}_{P,\mathrm{Ran}}.$$

When $\theta = 0$, we write:

$$\mathrm{Gr}^0_{P,\mathrm{Bun}_M} =: \mathrm{Gr}_{P,\mathrm{Bun}_M}.$$

We denote by j_P^{θ} the corresponding embedding:

$$j_P^{\theta} : \mathrm{Gr}^{\theta}_{P,\mathrm{Bun}_M} \hookrightarrow {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}}.$$

When $\theta = 0$, we write: $j_P^0 =: j_P$.

5.3.3. We have a Cartesian square

$$\begin{array}{ccc} \mathrm{Gr}_{P,\mathrm{Bun}_M} & \xrightarrow{j_P} & {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \\ {\scriptstyle \pi_P^0}\downarrow & & \downarrow{\scriptstyle \pi_{P,\mathrm{pol}}} \\ \mathrm{Bun}_P \times \mathrm{Ran} & \xrightarrow[j_{P,\mathrm{pol}}]{} & \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}. \end{array} \tag{5.3.1}$$

5.3.4. Note that the partially defined left adjoint $\pi^0_{P,!}$ of the functor $\pi_P^{0,!}$ is defined on $\omega_{\mathrm{Gr}_{P,\mathrm{Bun}_M}}$ by holonomicity of the dualizing sheaf. Moreover, by the universal homological contractibility statement of Proposition 5.2.8.2, the counit map

$$\pi^0_{P,!}(\omega_{\mathrm{Gr}_{P,\mathrm{Bun}_M}}) \to \omega_{\mathrm{Bun}_P \times \mathrm{Ran}} \tag{5.3.2}$$

is an isomorphism.

Since $\pi^!_{P,\mathrm{pol}}$ is $\mathrm{Rep}(\check{M} \times \check{G})_{\mathrm{Ran}}$-equivariant and by rigidity of the latter, the functor $\pi_{P,\mathrm{pol},!}$ is also $\mathrm{Rep}(\check{M} \times \check{G})_{\mathrm{Ran}}$-equivariant. We therefore get induced functors

$$\pi_{P,\mathrm{pol},!} : \mathrm{DrPl}_{\check{M},\check{G}}(D({}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}})) \longleftrightarrow \mathrm{DrPl}_{\check{M},\check{G}}(D(\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}})) : \pi^!_{P,\mathrm{pol}},$$

where the functor from left to right is only partially defined. We have similar assertions for enhanced Drinfeld-Plücker structures and Hecke structures.

5.3.5. Recall that we denote by $\mathbf{j}_!$ the !-extension of $\omega_{S^0_{P,\mathrm{Ran}}}$ to $\mathfrak{L}^+_{\mathrm{Ran}}\backslash\mathrm{Gr}_{G,\mathrm{Ran}}$. By Proposition 4.5.4.1, $\mathbf{j}_!$ is equipped with a canonical enhanced Drinfeld-Plücker structure. Since the projection

$$\mathrm{pr}_{\mathrm{Bun}_M} : {}_{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \longrightarrow \mathfrak{L}^+_{\mathrm{Ran}}M\backslash\,\mathrm{Gr}_{G,\mathrm{Ran}}$$

commutes with Hecke correspondences, the sheaf

$${}_{\mathrm{Bun}_M}\mathbf{j}_! := \mathrm{pr}^!_{\mathrm{Bun}_M}(\mathbf{j}_!)$$

is also equipped with a Drinfeld-Plücker structure. As a result, the sheaf

$${}_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} := \mathrm{pr}^!_{\mathrm{Bun}_M}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$$

comes with a canonical identification:

$${}_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \simeq \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}({}_{\mathrm{Bun}_M}\mathbf{j}_!).$$

5.3.6. Let us also denote by $\mathbf{j}^{\mathrm{glob}}_!$ the !-extension of $\omega_{\mathrm{Bun}_P \times \mathrm{Ran}}$ along $j_{P,\mathrm{pol}} : \mathrm{Bun}_P \times \mathrm{Ran} \to \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}$. By (5.3.2), we have:

$$\pi_{P,\mathrm{pol},!}({}_{\mathrm{Bun}_M}\mathbf{j}_!) \simeq \mathbf{j}^{\mathrm{glob}}_!. \tag{5.3.3}$$

By Proposition 4.5.4.1, $\mathbf{j}^{\mathrm{glob}}_!$ is equipped with an enhanced Drinfeld-Plücker structure.

The following lemma is immediate from universal homological contractibility of $\pi_{P,\mathrm{pol}}$ and $\mathrm{Rep}(\check{M}\times \check{G})_{\mathrm{Ran}}$-equivariance:

Lemma 5.3.6.1. *There is a canonical isomorphism:*

$${}_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} = \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}({}_{\mathrm{Bun}_M}\mathbf{j}_!) \simeq \pi^!_{P,\mathrm{pol}}(\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!)).$$

Moreover, the counit for $\pi^!_{P,\mathrm{pol}}$ applied to $\mathrm{Ind}^{\mathrm{Hecke}}_{\mathrm{DrPl}}(\mathbf{j}^{\mathrm{glob}}_!)$ gives an isomorphism:

$$\pi_{P,\mathrm{pol},!} \circ \pi^!_{P,\mathrm{pol}}(\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!)) \to \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!).$$

5.3.7. Let $i_{P,\mathrm{pol}}$ denote the closed embedding:

$$i_{P,\mathrm{pol}} : \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \to \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}.$$

Moreover, let $\mathrm{oblv}_{\mathrm{zer}}$ denote the forgetful map

$$\mathrm{oblv}_{\mathrm{zer}} : \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} \to \widetilde{\mathrm{Bun}}_{P}.$$

From the IC-sheaf on $\widetilde{\mathrm{Bun}}_P$ with respect to the perverse t-structure, we define the sheaf:

$$\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}} := i_{\mathrm{pol},*}(\mathrm{oblv}^!_{\mathrm{zer}}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})).$$

5.3.8. In what follows, we will use the notation $\dim(\mathrm{Bun}_P)$ to denote the locally constant function on Bun_P that takes the value

$$\dim(\mathrm{Bun}_M) + \dim(\mathrm{Bun}_{N_P}) + \langle 2(\rho_G - \rho_M), \eta\rangle$$

on the component of Bun_P living over the connected component Bun^η_M of Bun_M. We have the following local-to-global result:

Theorem 5.3.8.1. *There is a canonical isomorphism:*

$$_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \simeq \pi^!_{P,\mathrm{pol}}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}[\dim(\mathrm{Bun}_P)]).$$

Moreover, the counit morphism

$$\pi_{P,\mathrm{pol},!}(_{\mathrm{Bun}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}) \longrightarrow \mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}[\dim(\mathrm{Bun}_P)]$$

is an isomorphism.

Proof. By Lemma 5.3.6.1, it suffices to prove that there is a canonical identification:

$$\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!)[-\dim(\mathrm{Bun}_P)] \simeq \mathrm{IC}_{\widetilde{\mathrm{Bun}}_{P,\mathrm{pol}}}.$$

By Lemma (4.4.8.1) and (5.3.3), we get that $\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!)[-\dim(\mathrm{Bun}_P)]$ is supported on the image of the closed embedding $i_{P,\mathrm{pol}}$. We have a commutative diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{Gr}^\theta_{P,\mathrm{Bun}_M} & \longrightarrow & \widetilde{\mathrm{Gr}}_{P,\mathrm{Bun}_M} & \longrightarrow & _{\mathrm{Bun}_M}\mathrm{Gr}_{G,\mathrm{Ran}} \\
 & \swarrow & \downarrow & & \downarrow & & \downarrow \\
 & & _\theta\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} & \longrightarrow & \widetilde{\mathrm{Bun}}_{P,\mathrm{zer}} & \longrightarrow & \widetilde{\mathrm{Bun}}_{P,\mathrm{pol}} \\
 & & \downarrow & & \downarrow & & \\
\mathcal{H}^+_{M,X^\theta} & \longleftarrow & _\theta\widetilde{\mathrm{Bun}}_P & \longrightarrow & \widetilde{\mathrm{Bun}}_P & &
\end{array} \tag{5.3.4}$$

with all squares Cartesian and all vertical arrows universally homologically contractible. For the remainder of the proof, we will denote the !-pullback of $\mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{EnhDrPl}_{\check{M},\check{G}}}(\mathbf{j}^{\mathrm{glob}}_!)[-\dim(\mathrm{Bun}_P)]$ to $\widetilde{\mathrm{Bun}}_{P,\mathrm{zer}}$ by $\mathscr{I}$. We immediately note that $\mathscr{I}$ descends to a sheaf on $\widetilde{\mathrm{Bun}}_P$ by unitality, which we will also denote by $\mathscr{I}$. To complete the proof, it suffices to show that as a sheaf on $\widetilde{\mathrm{Bun}}_P$, the ! (resp. $*$) restrictions of $\mathscr{I}$ to the strata indexed by $\theta \neq 0$ lie in perverse cohomoloigcal degrees $\geqslant 1$ (resp. $\leqslant -1$), and that the restriction of $\mathscr{I}$ to Bun_P coincides with $\omega_{\mathrm{Bun}_P}[-\dim(\mathrm{Bun}_P)]$.

Step 1. Let us first compute the $*$-restriction of $\mathscr{I}$ to ${}_\theta\widetilde{\mathrm{Bun}}_P$. All sheaves in question are ind-holonomic, and hence by base change along the diagram (5.3.4), $\iota_\theta^*(\mathscr{I})$ is the !-pushforward of a sheaf $\mathcal{G}^\theta$ on $\mathrm{Gr}^\theta_{P,\mathrm{Bun}_M}$ to ${}_\theta\widetilde{\mathrm{Bun}}_P$. By Proposition 4.4.9.1, $\mathcal{G}^\theta$ has the property that it is the !-pullback of $\mathcal{O}(\check{N}_P)_{X^\theta}[-\langle 2(\rho_G-\rho_M),\theta\rangle - \dim(\mathrm{Bun}_P)]$ along the composition

$$\mathrm{Gr}^\theta_{P,\mathrm{Bun}_M} \to \mathcal{H}^+_{M,X^\theta} \to \mathfrak{L}^+_{X^\theta}M\backslash \mathrm{Gr}^+_{M,X^\theta}.$$

By universal homological contractibility of the map $\mathrm{Gr}^\theta_{P,\mathrm{Bun}_M} \to {}_\theta\widetilde{\mathrm{Bun}}_P$, it follows that $\iota_\theta^*(\mathscr{I})$ is the !-pullback of

$$\mathcal{O}(\check{N})_{X^\theta}[-\langle 2(\rho_G-\rho_M),\theta\rangle - \dim(\mathrm{Bun}_P)]$$

along the projection

$${}_\theta\widetilde{\mathrm{Bun}}_P \to \mathfrak{L}^+_{X^\theta}M\backslash \mathrm{Gr}^+_{M,X^\theta}. \tag{5.3.5}$$

We factor the above map as the composition:

$${}_\theta\widetilde{\mathrm{Bun}}_P \to \mathcal{H}^+_{M,X^\theta} \to \mathfrak{L}^+_{X^\theta}M\backslash \mathrm{Gr}^+_{M,X^\theta}.$$

Recall the t-structure from §3.4.17. A similar argument as in the proof of [Ber21, Lemma 2.1.15] shows that !-pullback along the map $\mathcal{H}^+_{M,X^\theta} \to \mathfrak{L}^+_{X^\theta}M\backslash \mathrm{Gr}^+_{M,X^\theta}$ is t-exact up to a cohomological shift by $\dim(\mathrm{Bun}_M)$.

Next, fix a component Bun^η_M of Bun_M and consider the stratum

$$\iota_\theta : {}_\theta\widetilde{\mathrm{Bun}}^\eta_P \to \widetilde{\mathrm{Bun}}^\eta_P.$$

By definition of the former, we have an isomorphism:

$${}_\theta\widetilde{\mathrm{Bun}}^\eta_P \xrightarrow{\sim} \mathcal{H}^+_{M,X^\theta} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}^{\theta+\eta}_P.$$

The projection

$${}_\theta\widetilde{\mathrm{Bun}}^\eta_P \simeq \mathcal{H}^+_{M,X^\theta} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}^{\theta+\eta}_P \longrightarrow \mathcal{H}^+_{M,X^\theta}$$

is smooth of relative dimension $\dim(\mathrm{Bun}_{N_P}) + \langle 2(\rho_G-\rho_M),\theta+\eta\rangle$, by [FGV01, Corollary 2.2.9] and the discussion in *loc. cit.* Moreover, for $\theta \neq 0$, the sheaf $\mathcal{O}(\check{N}_P)_{X^\theta}$ lives in perverse degrees $\leqslant -1$, cf. Lemma 3.4.18.2.

Putting the results together, it follows that for $\theta \neq 0$, the result of !-pulling the sheaf

$$\mathcal{O}(\check{N}_P)_{X^\theta}[\langle -2(\rho_G-\rho_M),\theta\rangle - \dim(\mathrm{Bun}_P)]$$

back to ${}_\theta\widetilde{\mathrm{Bun}}^\eta_P$ lives in perverse degrees at most

$$\begin{aligned} &-1-\dim(\mathrm{Bun}_M)-\dim(\mathrm{Bun}_{N_P})-\langle 2(\rho_G-\rho_M),\theta+\eta\rangle+\langle 2(\rho_G-\rho_M),\theta\rangle+\dim(\mathrm{Bun}_P) = \\ &\quad -1-\dim(\mathrm{Bun}_M)-\dim(\mathrm{Bun}_{N_P})-\langle 2(\rho_G-\rho_M),\theta+\eta\rangle+\langle 2(\rho_G-\rho_M),\theta\rangle \\ &\qquad +\dim(\mathrm{Bun}_M)+\dim(\mathrm{Bun}_{N_P})+\langle 2(\rho_G-\rho_M),\eta\rangle = -1. \end{aligned}$$

Step 2. For $\theta = 0$, the !-pullback of $\mathcal{O}(\check{N}_P)_{X^0}[-\dim(\mathrm{Bun}_P)]$ along (5.3.5) evidently gives $\omega_{\mathrm{Bun}_P}[-\dim(\mathrm{Bun}_P)]$.

Step 3. Using Lemma 4.5.7.1, a similar computation to Step 1 shows that the !-restriction of $\mathscr{I}$ along $\iota_\theta : {}_\theta\widetilde{\mathrm{Bun}}_P \to \widetilde{\mathrm{Bun}}_P$ lives in perverse degrees $\geqslant 1$. It follows that $\mathscr{I}$, when viewed as a sheaf on $\widetilde{\mathrm{Bun}}_P$, is canonically isomorphic to $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$, concluding the proof. □

Corollary 5.3.8.2. *We have canonical isomorphisms:*

$$\iota_\theta^!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \simeq \mathfrak{U}(\check{\mathfrak{n}}_P)_{X^\theta} \widetilde{\boxtimes} \mathrm{IC}_{\mathrm{Bun}_P};$$

$$\iota_\theta^*(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \simeq \mathcal{O}(\check{N}_P)_{X^\theta} \widetilde{\boxtimes} \mathrm{IC}_{\mathrm{Bun}_P}$$

as sheaves on ${}_\theta\widetilde{\mathrm{Bun}}_P \simeq \mathcal{H}^+_{M,X^\theta} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P$.

Proof. Using the second assertion of Theorem 5.3.8.1, we essentially proved that

$$\iota_\theta^*(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \simeq \mathcal{O}(\check{N}_P)_{X^\theta} \widetilde{\boxtimes} \mathrm{IC}_{\mathrm{Bun}_P}$$

in the course of the proof of *loc.cit.*[23]

By Verdier self-duality of $\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$ and (3.4.3), we get the assertion for $\iota_\theta^!(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P})$. □

[23]More precisely, we saw that we have an isomorphism $\iota_\theta^*(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \simeq \mathcal{O}(\check{N}_P)_{X^\theta} \widetilde{\boxtimes} \mathrm{IC}_{\mathrm{Bun}_P}$, up to a cohomological shift by some integer d. However, tracing through the proof of Theorem 5.3.8.1 and keeping track of the shifts, we see that $d = 0$.

6. Restriction of representations and the Casselman-Shalika formula

6.1. **Whittaker categories.** For a factorization category $\mathcal{C}$ with an action of the loop group $\mathfrak{L}_{\mathrm{Ran}}G$, we consider its *Whittaker category* $\mathrm{Whit}(\mathcal{C})$. When $\mathcal{C}$ is the category of D-modules on some prestack $\mathcal{Y}$, we will often use the notation $\mathrm{Whit}(\mathcal{Y}) := \mathrm{Whit}(D(\mathcal{Y}))$. In particular, we let $\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$ denote the *spherical Whittaker category*, i.e. the Whittaker category for D-modules on the affine Grassmannian.

6.1.1. The category $\mathrm{Whit}(\mathcal{C}) := \mathcal{C}^{\mathfrak{L}_{\mathrm{Ran}}N^-,\psi_{N^-,\mathrm{Ran}}}$ consists of objects in $\mathcal{C}$ that are equivariant for $\mathfrak{L}_{\mathrm{Ran}}N^-$ against a non-degenerate character $\psi_{N^-,\mathrm{Ran}} = \psi_{\mathfrak{L}_{\mathrm{Ran}}N^-}$ of $\mathfrak{L}_{\mathrm{Ran}}N^-$. In particular $\mathrm{Whit}(\mathcal{C})$ is equipped with a fully-faithful forgetful functor

$$\mathrm{Oblv}^{\mathrm{Whit}}_{\mathcal{C}} : \mathrm{Whit}(\mathcal{C}) \longrightarrow \mathcal{C}$$

admitting a (non-continuous) right adjoint. We will renormalize the right adjoint to be continuous in the case of $\mathcal{C} = D(\mathrm{Gr}_{G,\mathrm{Ran}})$ as follows. There are tautological equivalences:

$$\mathrm{Whit}(\mathfrak{L}_{\mathrm{Ran}}N^-) \simeq \mathrm{Whit}(\mathrm{Gr}_{N^-,\mathrm{Ran}}) \simeq D(\mathrm{Ran}).$$

We write $\psi_{N^-,\mathrm{Ran}}$ for the images of ω_{Ran} under these equivalences. We write $\psi_{G,\mathrm{Ran}}$ for the !-extension of $\psi_{N^-,\mathrm{Ran}} \in D(\mathrm{Gr}_{N^-,\mathrm{Ran}})$ along $\mathrm{Gr}_{N^-,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$.

Moreover, we write $\psi_{N^-,x}$ for the restriction of $\psi_{N^-,\mathrm{Ran}}$ to $x \in X$.

6.1.2. Now, convolution provides a pairing

$$- \underset{\mathfrak{L}_{\mathrm{Ran}}G}{\star} - : \mathrm{Whit}(\mathfrak{L}_{\mathrm{Ran}}G) \otimes D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

and for an object $\mathcal{F}$ of $D(\mathrm{Gr}_{G,\mathrm{Ran}})$, we write:

$$\mathrm{Av}_*^{\mathfrak{L}_{\mathrm{Ran}}N^-,\psi}(\mathcal{F}) := \psi_{G,\mathrm{Ran}} \underset{\mathfrak{L}_{\mathrm{Ran}}G}{\star} \mathcal{F}.$$

We call the functor $\mathrm{Av}_*^{\mathfrak{L}_{\mathrm{Ran}}(N^-),\psi} : D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$ *renormalized Whittaker averaging.*

6.1.3. Denote by $\mathrm{Whit}(\mathrm{Gr}_{G,x})$ the fiber of $\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$ at a k-point x of X. Then $\mathrm{Whit}(\mathrm{Gr}_{G,x})$ has a set of compact generators ψ_μ where μ ranges over anti-dominant coweights of G. Specifically, for some anti-dominant μ, the sheaf ψ_μ is given by !-extending the object corresponding to the unit under the evident equivalence

$$\mathrm{Whit}(S_x^{-,\mu}) = D(S_x^{-,\mu})^{\mathfrak{L}_x N^-,\psi_{N^-,x}} \simeq \mathsf{Vect},$$

where $S_x^{-,\mu}$ denotes the μ-th semi-infinite orbit for N^- in $\mathrm{Gr}_{G,x}$. By the cleanness theorem of [FGV01], ψ_μ is equivalently the $*$-extension from $S_x^{-,\mu}$.

6.1.4. Define a t-structure on $\mathrm{Whit}(\mathrm{Gr}_{G,x})$ by declaring an object $\mathcal{F}$ to lie in $\mathrm{Whit}(\mathrm{Gr}_{G,x})^{\leqslant 0}$ if an only if

$$H^0\,\mathrm{Hom}_{\mathrm{Whit}(\mathrm{Gr}_{G,x})}(\mathcal{F}, \psi_\mu[-k]) = 0$$

for all anti-dominant μ for all integers $k > 0$.

We have the following geometric Casselman-Shalika formula (see [FGV01] and [Ras18]):

Theorem 6.1.4.1. *There is a canonical equivalence of* $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ *factorization module categories*

$$\mathrm{CS}_G : \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

The !-fiber at any k-point $x \in X$ gives a t-exact equivalence

$$\mathrm{CS}_{G,x} : \mathrm{Whit}(\mathrm{Gr}_{G,x}) \xrightarrow{\sim} \mathrm{Rep}(\check{G})$$

that sends ψ_μ to the irreducible representation of highest weight $w_0(\mu)$.

6.2. **The Jacquet functors.** In this section we will construct several functors

$$\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \longrightarrow \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}),$$

where the Whittaker condition on $D(\mathrm{Gr}_{M,\mathrm{Ran}})$ is understood to be relative to $\mathfrak{L}_{\mathrm{Ran}}N_M^-$ via the non-degenerate character $\psi_{N^-,\mathrm{Ran}}$ obtained by restricting the same named character along the inclusion $\mathfrak{L}_{\mathrm{Ran}}N_M^- \hookrightarrow \mathfrak{L}_{\mathrm{Ran}}N^-$.

6.2.1. Consider the prestack

$${}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}} := \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}}\,.$$

Unwinding the definitions, we see there is a canonical map

$${}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}} \longrightarrow \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \tag{6.2.1}$$

which is easily seen to be an isomorphism. Hence we have an equivalence of categories:

$$D({}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathrm{Ran})}(D(\mathrm{Gr}_{G,\mathrm{Ran}}), D(\mathrm{Gr}_{M,\mathrm{Ran}})). \tag{6.2.2}$$

That is, we can view D-modules on ${}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}}$ as kernels of functors $D(\mathrm{Gr}_{G,\mathrm{Ran}}) \to D(\mathrm{Gr}_{M,\mathrm{Ran}})$ relative to the action of $D(\mathrm{Ran})$ on both sides.

6.2.2. Note we have a diagonal action of $\mathfrak{L}_{\mathrm{Ran}}P$ on $\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}}$ and we have an isomorphism

$$\mathfrak{L}^+_{\mathrm{Ran}}N_P\mathfrak{L}_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}} \xrightarrow{\sim} \mathfrak{L}_{\mathrm{Ran}}P\backslash(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}})$$

such that the diagram

$$\begin{array}{ccc}
{}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}} & \longrightarrow & \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \\
\downarrow & & \downarrow \\
\mathfrak{L}^+_{\mathrm{Ran}}M\mathfrak{L}_{\mathrm{Ran}}N_P\backslash \mathrm{Gr}_{G,\mathrm{Ran}} & \longrightarrow & \mathfrak{L}_{\mathrm{Ran}}P\backslash(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}})
\end{array}$$

commutes. Here the left vertical map is the projection to $\mathfrak{L}^+_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}}$ followed by the canonical map to $\mathfrak{L}^+_{\mathrm{Ran}}M\mathfrak{L}_{\mathrm{Ran}}N_P\backslash \mathrm{Gr}_{G,\mathrm{Ran}}$, and the right vertical map is the quotient map.

It follows that we obtain a canonical equivalence:

$$\mathrm{SI}_{P,\mathrm{Ran}} := D(\mathfrak{L}^+_{\mathrm{Ran}}N_P\mathfrak{L}_{\mathrm{Ran}}M\backslash \mathrm{Gr}_{G,\mathrm{Ran}}) \simeq \mathrm{Hom}_{D(\mathfrak{L}_{\mathrm{Ran}}P)}(D(\mathrm{Gr}_{G,\mathrm{Ran}}), D(\mathrm{Gr}_{M,\mathrm{Ran}})).$$

That is, we can view objects of the semi-infinite category as as $\mathfrak{L}_{\mathrm{Ran}}P$-equivariant functors from $D(\mathrm{Gr}_{G,\mathrm{Ran}})$ to $D(\mathrm{Gr}_{M,\mathrm{Ran}})$.

6.2.3. We therefore obtain a canonical pairing

$$D(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{D(\mathrm{Ran})}{\otimes} \mathrm{SI}_{P,\mathrm{Ran}} \longrightarrow D(\mathrm{Gr}_{M,\mathrm{Ran}}).$$

By forgetting from $\mathfrak{L}_{\mathrm{Ran}}N^-$ equivariance against ψ to $\mathfrak{L}_{\mathrm{Ran}}N_M^-$ equivariance against the restriction of χ to the latter, we obtain a pairing:

$$\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{D(\mathrm{Ran})}{\otimes} \mathrm{SI}_{P,\mathrm{Ran}} \longrightarrow \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}). \tag{6.2.3}$$

In particular, from the objects $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}, \mathbf{j}_! \in \mathrm{SI}_{P,\mathrm{Ran}}$, we get functors

$$\mathrm{Jac}^M_{!*},\ \mathrm{Jac}^M_! : \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \longrightarrow \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}),$$

respectively.

6.2.4. Since $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is an object of $\mathrm{Hecke}_{\check{M},\check{G}}(\mathrm{SI}_{P,\mathrm{Ran}})$ by definition, we get that $\mathrm{Jac}^M_{!*}$ is $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear. That is:

$$\mathrm{Jac}^M_{!*}(V \star -) \simeq \mathrm{Res}^{\check{G}}_{\check{M}}(V) \star \mathrm{Jac}^M_{!*}(-), \quad V \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

Similarly, from the enhanced Drinfeld-Plücker structure on $\mathbf{j}_!$, we get:

$$\mathrm{Jac}^M_!(V \star -) \simeq C^\bullet(\check{\mathfrak{n}}_P, V) \underset{\Omega(\check{\mathfrak{n}}_P)_{\mathrm{Ran}}}{\star} \mathrm{Jac}^M_!(-), \quad V \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

6.3. **Composing Jacquet functors.** We have Jacquet functors

$$\mathrm{Jac}^M_{!*} : \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \longrightarrow \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}});$$

$$\mathrm{Jac}^{T_M}_{!*} : \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}) \longrightarrow D(\mathrm{Gr}_{T,\mathrm{Ran}})$$

associated to the parabolics P and $B_M := B \cap M$, respectively. In the notation, we denote by T_M the maximal torus $T \subseteq M$ to emphasize that the domain of $\mathrm{Jac}^{T_M}_{!*}$ is the Whittaker category of M. In this section we will show how to compose Jacquet functors for G and M.

6.3.1. In what follows, denote by $\mathrm{Gr}_{P,\mathrm{Ran}}$ the fiber product

$$\mathrm{Gr}_{P,\mathrm{Ran}} := \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \backslash \mathfrak{L}^+_{\mathrm{Ran}} M S^0_{\mathrm{Ran}}.$$

Similarly, define:

$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} := \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \widetilde{S}^0_{\mathrm{Ran}}.$$

We will also denote by ${}_{\mathrm{Gr}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ the !-pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to $\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}$ along the projection:

$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \longrightarrow \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \widetilde{S}^0_{\mathrm{Ran}}.$$

We have the following theorem.

Theorem 6.3.1.1. *The composition* $\mathrm{Jac}^{T_M}_{!*} \circ \mathrm{Jac}^M_{!*}$ *is canonically equivalent to the Jacquet functor*

$$\mathrm{Jac}^T_{!*} : \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to D(\mathrm{Gr}_{T,\mathrm{Ran}})$$

associated to the Borel subgroup B of G.

Proof.

Step 1. We will consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} \overline{\mathrm{Gr}}_{B_M,\mathrm{Ran}} & & & & \\
 & & & \swarrow^{\tilde{\mathfrak{p}}_{P,B_M,\mathrm{Ran}}} & & \searrow^{\tilde{\mathfrak{q}}_{P,B_M,\mathrm{Ran}}} & & & \\
 & & \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} & & & & \overline{\mathrm{Gr}}_{B_M,\mathrm{Ran}} & & \\
 & \swarrow^{\tilde{\mathfrak{p}}_{P,\mathrm{Ran}}} & & \searrow^{\tilde{\mathfrak{q}}_{P,\mathrm{Ran}}} & & \swarrow^{\bar{\mathfrak{p}}_{B_M,\mathrm{Ran}}} & & \searrow^{\bar{\mathfrak{q}}_{B_M,\mathrm{Ran}}} & \\
\mathrm{Gr}_{G,\mathrm{Ran}} & & & & \mathrm{Gr}_{M,\mathrm{Ran}} & & & & \mathrm{Gr}_{T,\mathrm{Ran}}\,.
\end{array}$$

Base change and the projection formula shows that the functor $\mathrm{Jac}^{T_M}_{!*} \circ \mathrm{Jac}^{M}_{!*}$ is given by

$$\mathcal{F} \longmapsto (\overline{\mathfrak{q}}_{B_M,\mathrm{Ran}} \circ \widetilde{\mathfrak{q}}_{P,B_M,\mathrm{Ran}})_*((\widetilde{\mathfrak{p}}_{P,\mathrm{Ran}} \circ \widetilde{\mathfrak{p}}_{P,B_M,\mathrm{Ran}})^!(\mathcal{F}) \overset{!}{\otimes} {}_{\mathrm{Gr}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \widetilde{\boxtimes}_{\mathrm{Gr}_T} \mathrm{IC}^{\frac{\infty}{2}}_{B_M,\mathrm{Ran}}).$$

By composing Plücker data and forgetting the M-bundle, we obtain a map

$$\mathfrak{r} : \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} \overline{\mathrm{Gr}}_{B_M,\mathrm{Ran}} \longrightarrow \overline{\mathrm{Gr}}_{B,\mathrm{Ran}},$$

which is an equivalence on $\mathrm{Gr}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} \mathrm{Gr}_{B_M,\mathrm{Ran}} \simeq \mathrm{Gr}_{B,\mathrm{Ran}}$ and such that

$$\overline{\mathfrak{q}}_{B_M,\mathrm{Ran}} \circ \widetilde{\mathfrak{q}}_{P,B_M,\mathrm{Ran}} = \overline{\mathfrak{q}}_{B,\mathrm{Ran}} \circ \mathfrak{r} \text{ and } \widetilde{\mathfrak{p}}_{P,\mathrm{Ran}} \circ \widetilde{\mathfrak{p}}_{P,B_M,\mathrm{Ran}} = \overline{\mathfrak{p}}_{B,\mathrm{Ran}} \circ \mathfrak{r}.$$

Another application of the projection formula shows that the composition $\mathrm{Jac}^{T_M}_{!*} \circ \mathrm{Jac}^{M}_{!*}$ is given by

$$\mathcal{F} \longmapsto \overline{\mathfrak{q}}_{B,\mathrm{Ran},*}(\overline{\mathfrak{p}}^!_{B,\mathrm{Ran}}(\mathcal{F}) \overset{!}{\otimes} \mathfrak{r}_*({}_{\mathrm{Gr}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \widetilde{\boxtimes}_{\mathrm{Gr}_T} \mathrm{IC}^{\frac{\infty}{2}}_{B_M,\mathrm{Ran}})).$$

To conclude the proof, it therefore suffices to produce a canonical isomorphism

$${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}} \xrightarrow{\sim} \mathfrak{r}_*({}_{\mathrm{Gr}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \widetilde{\boxtimes}_{\mathrm{Gr}_T} \mathrm{IC}^{\frac{\infty}{2}}_{B_M,\mathrm{Ran}}).$$

Step 2. To this end, note we have closed embeddings

$$\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \hookrightarrow {}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}}$$

and

$$\overline{\mathrm{Gr}}_{B_M,\mathrm{Ran}} \hookrightarrow {}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{Gr}_{M,\mathrm{Ran}}.$$

Hence we can identify ${}_{\mathrm{Gr}_M}\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ and ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{B_M,\mathrm{Ran}}$ with their pushforwards along the above maps. Let us further recall that by (6.2.1) we have isomorphisms

$${}_{\mathrm{Gr}_{M,\mathrm{Ran}}}\mathrm{Gr}_{G,\mathrm{Ran}} \simeq \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}}$$

and

$${}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{Gr}_{M,\mathrm{Ran}} \simeq \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}}$$

that commute with Hecke correspondences for G and M in the first case, and for M and T in the second.

By the definition of the semi-infinite IC sheaf (see §4.3.7), we have:

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} := \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}})$$

A similar equation holds for the pair (M, B_M).

It follows that the pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to $\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}}$ is the Hecke object induced from a Drinfeld-Plücker structure on the delta sheaf ${}_{\mathrm{Gr}_M}\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$ defined as pushforward of the dualizing sheaf along the closed embedding

$$\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M \underset{\mathbb{B}\mathfrak{L}^+_{\mathrm{Ran}}M}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \simeq \mathrm{Gr}_{M,\mathrm{Ran}} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}} \times_{\mathrm{Ran}} \mathrm{Gr}_{M,\mathrm{Ran}},$$

and similarly for ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{B_M,\mathrm{Ran}}$.

The projection map

$$p_{G,T} : \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}} \longrightarrow \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}}$$

onto the first and third factors commutes with Hecke functors for G and T. It follows that the functor

$$p_{G,T,*} : D(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}}) \longrightarrow D(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}})$$

commutes with the action of the category $\mathrm{Rep}(\check{G} \times \check{M} \times \check{T})_{\mathrm{Ran}}$, where the latter acts on

$$D(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}})$$

through the restriction functor

$$\mathrm{Rep}(\check{G} \times \check{M} \times \check{T})_{\mathrm{Ran}} \longrightarrow \mathrm{Rep}(\check{G} \times \check{T})_{\mathrm{Ran}}$$

given by forgetting the $\check{M}$ module structure.

Step 3. We consider

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \otimes \mathcal{O}(\overline{\check{N}_M\backslash\check{M}})_{\mathrm{Ran}} := (\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \otimes \mathcal{O}(\overline{\check{N}_M\backslash\check{M}}))_{\mathrm{Ran}}$$

as an object of $\mathrm{Rep}(\check{G} \times \check{M} \times \check{T})_{\mathrm{Ran}}$ via the diagonal action of $\check{M}$

The $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}}$ module structure on ${}_{\mathrm{Gr}_M}\delta_{\mathrm{Gr}_G,\mathrm{Ran}}$ and the $\mathcal{O}(\overline{\check{N}_M\backslash\check{M}})_{\mathrm{Ran}}$ module structure on ${}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_M,\mathrm{Ran}}$ give an action of

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \otimes \mathcal{O}(\overline{\check{N}_M\backslash\check{M}})_{\mathrm{Ran}}$$

on the sheaf

$${}_{\mathrm{Gr}_M}\delta_{\mathrm{Gr}_G,\mathrm{Ran}} \overset{!}{\otimes} {}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_M,\mathrm{Ran}} \in D(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{M,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}}),$$

and therefore a similar action on

$$p_{G,T,*}({}_{\mathrm{Gr}_M}\delta_{\mathrm{Gr}_G,\mathrm{Ran}} \otimes {}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_M,\mathrm{Ran}}) \simeq {}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}.$$

Note that multiplication gives a $\check{G} \times \check{T}$ equivariant morphism

$$\check{N}_P\backslash\check{G} \times \check{N}_M\backslash\check{M} \longrightarrow \check{N}\backslash\check{G}.$$

Hence we get a map

$$\mathcal{O}(\overline{\check{N}\backslash\check{G}})_{\mathrm{Ran}} \longrightarrow \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \otimes \mathcal{O}(\overline{\check{N}_M\backslash\check{M}})_{\mathrm{Ran}} \tag{6.3.1}$$

in $\mathrm{Rep}(\check{G} \times \check{M} \times \check{T})_{\mathrm{Ran}}$, where we consider $\check{M}$ as acting trivially on $\mathcal{O}(\check{N}\backslash\check{G})$.

Restricting along the morphism (6.3.1), we view ${}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$ as a module for $\mathcal{O}(\overline{\check{N}\backslash\check{G}})$; i.e., as a Drinfeld-Plücker object.

In order to conclude the proof, it is therefore enough to show that the above Drinfeld-Plücker structure on ${}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$ coincides with the Drinfeld-Plücker structure constructed in Appendix (B). The first step towards this is to note the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{\mathrm{Ran}} \otimes \mathcal{O}(\overline{\check{N}_M\backslash\check{M}})_{\mathrm{Ran}}\text{-mod}(D_{G,T}) & \longrightarrow & \mathcal{O}(\check{G})_{\mathrm{Ran}} \otimes \mathcal{O}(\check{G})_{\mathrm{Ran}}\text{-mod}(D_{G,T}) \\ \downarrow & & \downarrow \\ \mathcal{O}(\overline{\check{N}\backslash\check{G}})_{\mathrm{Ran}}\text{-mod}(D_{G,T}) & \longrightarrow & \mathcal{O}(\check{G})_{\mathrm{Ran}}\text{-mod}(D_{G,T}), \end{array}$$

where we have used the notation

$$D_{G,T} := D(\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{Ran}}).$$

Here, the horizontal maps are the induction functors from a Drinfeld-Plücker structure to a Hecke structure (see §4.2), and the vertical maps are given by restriction. Commutativity of this diagram follows in turn from the fact that the square

$$\begin{array}{ccc} \check{G} \times \check{G} & \longrightarrow & \check{G} \\ \downarrow & & \downarrow \\ \check{N}_P \backslash \check{G} \times \check{N}_M \backslash \check{M} & \longrightarrow & \check{G}/\check{N} \end{array}$$

is Cartesian.

Step 4. To identify the Drinfeld-Plücker structures on ${}_{\mathrm{Gr}_T}\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$, we note that it suffices to identify them over each power X^I of the curve in a compatible way. For simplicity, we identify them when further restricted to a point $x \in X$.[24]

Let $\mathcal{C}$ be a category acted on by $\mathrm{Rep}(\check{G} \times \check{M} \times \check{T})$. For $H \in \{\check{G}, \check{M}, \check{T}\}$, denote by $\underset{H}{\star}$ the corresponding action of $\mathrm{Rep}(H)$ on $\mathcal{C}$. An action of the algebra $\mathcal{O}(\overline{\check{N}_P \backslash \check{G}}) \otimes \mathcal{O}(\overline{\check{N}_M \backslash \check{M}})$ on an object c of $\mathcal{C}$ amounts to the data of, for every $V \in \mathrm{Rep}(\check{G})$, maps

$$V^{\check{N}_P} \underset{\check{M}}{\star} c \longrightarrow V \underset{\check{G}}{\star} c$$

satisfying the Plücker compatibilities, together with, for every representation $W \in \mathrm{Rep}(\check{M})$, maps

$$W^{\check{T}} \underset{\check{T}}{\star} c \longrightarrow W \underset{\check{M}}{\star} c$$

satisfying the Plücker compatibilities.
Restricting to an action of $\mathcal{O}(\overline{\check{N} \backslash \check{G}})$ amounts to taking the composition of the maps:

$$V^{\check{N}} \underset{\check{T}}{\star} c = (V^{\check{N}_P})^{\check{N}_M} \underset{\check{T}}{\star} c \longrightarrow V^{\check{N}_P} \underset{\check{M}}{\star} c \longrightarrow V \underset{\check{G}}{\star} c.$$

The latter assertion can be seen by unwinding the definitions and from the claim that the map (6.3.1) identifies $\mathcal{O}(\overline{\check{N}_P \backslash \check{G}})$ with $\check{M}$ invariants in $\mathcal{O}(\overline{\check{N}_P \backslash \check{G}}) \otimes \mathcal{O}(\overline{\check{N}_M \backslash \check{M}})$.

Lastly, it remains to check that the maps in the Drinfeld-Plücker structures at a point from §B.0.4 for the pairs $(\check{G}, \check{P})$ and $(\check{M}, \check{B}_M)$ compose to the Drinfeld-Plücker structure for the pair $(\check{G}, \check{B})$, but this is clear. □

6.4. **The !∗-Jacquet functor and restriction.** Let $\mathrm{Res}^{\check{G}}_{\check{M}}$ denote the symmetric monoidal functor

$$\mathrm{Res}^{\check{G}}_{\check{M}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

obtained from the symmetric monoidal functor of restriction of representations via twisted arrows.

Moreover, denote by $C^\bullet(\check{\mathfrak{n}}_P, -)$ the functor

$$C^\bullet(\check{\mathfrak{n}}_P, -) : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

given by restricting along $\check{P} \to \check{G}$ and taking Lie algebra cohomology against $\check{\mathfrak{n}}_P$, cf. §4.2.6.

[24] As remarked in the footnote in §B.0.8, all sheaves involved are in the heart of the relative perverse t-structure defined in [HS23], and hence it suffices to identify the $\mathcal{O}(\overline{\check{G}/\check{N}})_{\mathrm{Ran}}$-module structures at a point.

6.4.1. In this subsection we will prove the following theorem.

Theorem 6.4.1.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{Jac}^M_{!*}} & \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}) \\ {\scriptstyle \mathrm{CS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{CS}_M} \\ \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow[\mathrm{Res}^{\check{G}}_{\check{M}}]{} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}}. \end{array}$$

By Koszul duality we obtain the following as a corollary to the above theorem.

Corollary 6.4.1.2. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{Jac}^M_{!}} & \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}) \\ {\scriptstyle \mathrm{CS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{CS}_M} \\ \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow[C^\bullet(\check{\mathfrak{n}}_P,-)]{} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}}. \end{array}$$

Proof. Since $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ has a coaction of $\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}$, so does the functor $\mathrm{Jac}^M_{!*}$.

The functor $\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}}(\mathrm{Jac}^M_{!*})$ given by post-composing with the functor of taking $\mathcal{O}(\check{N}_P)$-invariants is therefore given by the kernel $\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}}(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$ under the pairing (6.2.3). Now the result follows from Theorem 6.4.1.1 and Proposition 4.5.4.1, observing that we have an isomorphism of functors:

$$\mathrm{Inv}_{\mathcal{O}(\check{N}_P)_{\mathrm{Ran}}} \circ \mathrm{Res}^{\check{G}}_{\check{M}} \simeq C^\bullet(\check{\mathfrak{n}}_P, -).$$

□

6.4.2. It remains to prove Theorem 6.4.1.1. We will reduce this theorem to a certain vanishing assertion when $P = B$ where we can invoke results of Raskin [Ras21].

Consider the commutative diagram:

$$\begin{array}{ccccccc} & & & \widetilde{\mathrm{Gr}}_{P,B^-,B^-_M,\mathrm{Ran}} & & & \\ & & \swarrow & & \searrow & & \\ & S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} & & & & \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} S^{-,0}_{B_M,\mathrm{Ran}} & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ S^{-,0}_{B,\mathrm{Ran}} & & & \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} & & & S^{-,0}_{B_M,\mathrm{Ran}} \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & \mathrm{Gr}_{G,\mathrm{Ran}} & & & & \mathrm{Gr}_{M,\mathrm{Ran}} & \end{array}$$

all of whose squares are Cartesian. For convenience, we have denoted the fiber product

$$S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} S^{-,0}_{B_M,\mathrm{Ran}}$$

by $\widetilde{\mathrm{Gr}}_{P,B^-,B^-_M,\mathrm{Ran}}$.

6.4.3. We have the vacuum Whittaker object $\psi_{G,\mathrm{Ran}}$, which is cleanly extended to $\mathrm{Gr}_{G,\mathrm{Ran}}$ from $S^{-,0}_{B,\mathrm{Ran}} = \mathrm{Gr}_{N^-,\mathrm{Ran}}$. We would like to understand the sheaf

$$\mathscr{F}_\psi := \iota^!_{B_M}(\mathrm{Jac}^M_{!*}(\psi_{G,\mathrm{Ran}})) \tag{6.4.1}$$

as an object of the category $\mathrm{Whit}(S^{-,0}_{B_M,\mathrm{Ran}})$. Here ι_{B_M} denotes the inclusion $S^{-,0}_{B_M,\mathrm{Ran}} \to \mathrm{Gr}_{M,\mathrm{Ran}}$.

We claim that in the diagram above, we can replace $\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}$ with the open locus

$$\mathrm{Gr}_{P,\mathrm{Ran}} \subseteq \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}.$$

That is, we claim that the open embedding

$$\mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}} := S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \mathrm{Gr}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} S^{-,0}_{B_M,\mathrm{Ran}} \hookrightarrow \widetilde{\mathrm{Gr}}_{P,B^-,B^-_M,\mathrm{Ran}}$$

is an isomorphism. Indeed, that we are taking the fiber over $S^{-,0}_{B_M,\mathrm{Ran}}$ forces the P-reduction over the disc to be non-degenerate. In more detail, the data of an S-point of the fiber product

$$S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}$$

is the data, for every irreducible representation V of G, of a point $(x_I, \mathcal{P}_G, \alpha)$ of $\mathrm{Gr}_{G,\mathrm{Ran}}$ and a point $(x_I, \mathcal{P}_M, \mathcal{P}_G)$ of $\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}$ such that the corresponding meromorphic maps

$$V^{N_P}_{\mathcal{P}_M} \longrightarrow V_{\mathcal{P}_G} \longrightarrow \mathcal{O}_{X\times S} = V^{N^-_B}_{\mathcal{P}^0_T}$$

are regular and the second map is non-degenerate. Note we are assuming V is irreducible for the last equality. Requiring that the corresponding point of $\mathrm{Gr}_{M,\mathrm{Ran}}$ lands in $S^{-,0}_{B_M,\mathrm{Ran}}$ is the condition that the composition of the above maps is non-degenerate for every representation V. Whenever V^{N_P} is one-dimensional, non-degeneracy of the composition implies non-degeneracy of the first map. Moreover, non-degeneracy of the generalized P-reduction can be checked using only representations V such that V^{N_P} is one-dimensional.

It follows that $\mathscr{F}_\psi$ can be computed by base-change along the above diagram, replacing $\widetilde{\mathrm{Gr}}_{P,\mathrm{Ran}}$ by $\mathrm{Gr}_{P,\mathrm{Ran}}$.

Lemma 6.4.3.1. *The composition*

$$\mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}} \to \mathrm{Gr}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} S^{-,0}_{B_M,\mathrm{Ran}} \to S^{-,0}_{B_M,\mathrm{Ran}}$$

is an isomorphism.

Proof. The group $\mathfrak{L}_{\mathrm{Ran}} N^-_M$ acts on $\mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}}$ in such a way that the projection

$$\mathrm{pr}_{B_M} : \mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}} \to S^{-,0}_{B_M,\mathrm{Ran}}$$

is equivariant for the usual action of $\mathfrak{L}_{\mathrm{Ran}} N^-_M$ on $S^{-,0}_{B_M,\mathrm{Ran}}$. Since the latter action is transitive, it follows that pr_{B_M} is an fppf locally trivial fiber bundle with typical fiber

$$\mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}} \underset{S^{-,0}_{B_M,\mathrm{Ran}}}{\times} \mathrm{Ran} \simeq S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \mathrm{Gr}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} \mathrm{Ran}$$

where the map $\mathrm{Ran} \to S^{-,0}_{B_M,\mathrm{Ran}}$ is the canonical section to the factorization morphism.

We have

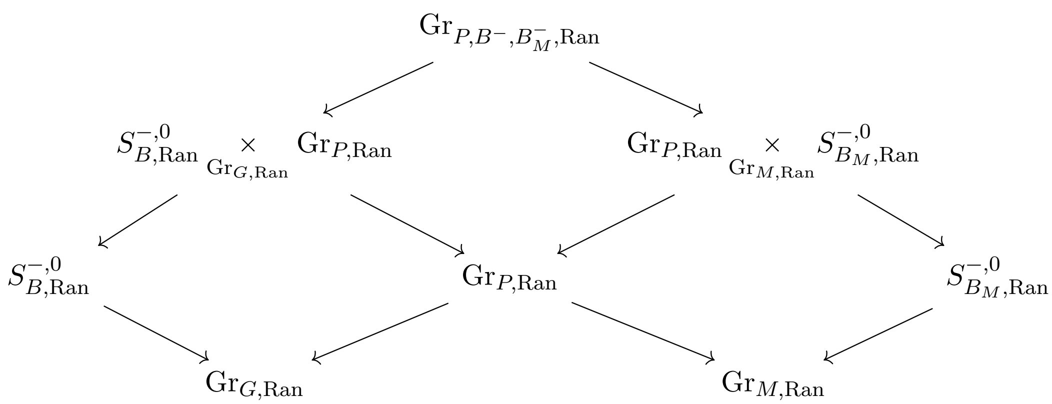
$$S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} \mathrm{Gr}_{P,\mathrm{Ran}} \underset{\mathrm{Gr}_{M,\mathrm{Ran}}}{\times} \mathrm{Ran} \simeq S^{-,0}_{B,\mathrm{Ran}} \underset{\mathrm{Gr}_{G,\mathrm{Ran}}}{\times} S^0_{B,\mathrm{Ran}} \simeq \mathrm{Ran},$$

and the result follows. ∎

Corollary 6.4.3.2. *There is a canonical isomorphism:*

$$\mathscr{F}_\psi \xrightarrow{\sim} \iota^!_{B_M}(\psi_{M,\mathrm{Ran}})$$

Proof. This follows by diagram chase along



using base change, taking into consideration the equivalence

$$D(S^{-,0}_{B_M,\mathrm{Ran}}) \xrightarrow{\sim} D(\mathrm{Gr}_{P,B^-,B^-_M,\mathrm{Ran}})$$

afforded by Lemma 6.4.3.1. □

6.4.4. Consider the counit map:

$$\psi_{M,\mathrm{Ran}} \simeq \iota_{B_M,!} \circ \iota^!_{B_M}(\mathrm{Jac}^M_{!*}(\psi_{G,\mathrm{Ran}})) \to \mathrm{Jac}^M_{!*}(\psi_{G,\mathrm{Ran}}). \tag{6.4.2}$$

Here the first isomorphism comes from Corollary 6.4.3.2. In the remaining part of this subsection, we will prove:

Theorem 6.4.4.1. *The counit map (6.4.2) is an isomorphism.*

We start by noting that the above theorem is sufficient for our purposes:

Lemma 6.4.4.2. *Suppose the counit map (6.4.2) is an isomorphism. Then the assertion of Theorem 6.4.1.1 holds.*

Proof. Recall that the Hecke structure on the semi-infinite IC sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ gives $\mathrm{Jac}^M_{!*}$ the structure of a $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ linear functor, see §6.2.4.

By the Casselman-Shalika formula for G and M, we have

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}(\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}), \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}})) \simeq \mathrm{Hom}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}(\mathrm{Rep}(\check{G})_{\mathrm{Ran}}, \mathrm{Rep}(\check{M})_{\mathrm{Ran}}), \tag{6.4.3}$$

while the right-hand side is nothing other than $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$. It follows that any $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear functor $\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}})$ is determined by where it sends $\psi_{G,\mathrm{Ran}}$. Thus, we may check that $\mathrm{Jac}^M_{!*}$ and $\mathrm{Res}^{\check{G}}_{\check{M}}$ coincide by checking that they both send $\psi_{G,\mathrm{Ran}}$ to $\psi_{M,\mathrm{Ran}}$. □

6.4.5. Hence we need to prove Theorem 6.4.4.1. To do this, we will use Theorem 6.3.1.1 to reduce to the principal case; i.e., when $P = B$. As such, let us assume that Theorem 6.4.4.1 holds for an arbitrary reductive group (e.g., G or M) whenever the corresponding Levi is a torus. Under this assumption, let us prove Theorem 6.4.4.1 for an arbitrary Levi.

We write $\mathrm{Jac}^{T_M}_{!*} : \mathrm{Whit}(\mathrm{Gr}_{M,\mathrm{Ran}}) \to \mathrm{Whit}(\mathrm{Gr}_{T,\mathrm{Ran}})$ for the !∗-Jacquet functor from the Levi M to the maximal torus T. It suffices to check that (6.4.2) is an isomorphism after applying the functor

$\mathrm{Jac}^{T_M}_{!*}$. Indeed, having assumed Theorem 6.4.4.1 is true in the principal case, Lemma 6.4.4.2 implies that Theorem 6.4.1.1 is true in the principal case. This in turn implies that $\mathrm{Jac}^{T_M}_{!*}$ is conservative.

However, by Theorem 6.3.1.1 we have:

$$\mathrm{Jac}^{T_M}_{!*}(\mathrm{Jac}^{M}_{!*}(\psi_{G,\mathrm{Ran}})) \simeq \mathrm{Jac}^{T}_{!*}(\psi_{G,\mathrm{Ran}}) \simeq \delta_{\mathrm{Gr}_{T,\mathrm{Ran}}} \simeq \mathrm{Jac}^{T_M}_{!*}(\psi_{M,\mathrm{Ran}}).$$

Hence we are reduced to showing that Theorem 6.4.4.1 holds in the principal case.

6.4.6. We will produce two proofs in the principal case. The first argument uses a result of Raskin from [Ras21], and the second uses techniques from [GL19]. Since the first argument will be rigorous, we will allow ourselves to only give a sketch of the second.

The reason we will not provide full details for the second argument is that in [GL19] the authors work in a twisted setting with a non-degeneracy assumption on the twisting. As such, we cannot cite their results directly. However, one can show that their arguments can be adapted to the non-twisted setting and that the main difficulty in doing so is notational. Although we allow ourselves to forego complete rigor, we still believe an indication of the second proof is valuable in that it avoids the use of perverse t-structures.

First proof of Theorem 6.4.4.1 in the principal case. We need to show that for any k-point

$$a : \mathrm{pt} \longrightarrow \mathrm{Gr}_{T,\mathrm{Ran}}$$

that does not lie in the image of the unit section

$$\mathrm{Ran} \longrightarrow \mathrm{Gr}_{T,\mathrm{Ran}},$$

the vector space $a^!(\mathrm{Jac}^{T}_{!*}(\psi_{G,\mathrm{Ran}}))$ vanishes. Since we are in the principal case, we write $\overline{\mathrm{Gr}}_{B,\mathrm{Ran}}$ instead of $\widetilde{\mathrm{Gr}}_{B,\mathrm{Ran}}$.

Consider the diagram

$$\begin{array}{ccccc}
 & & \overline{\mathrm{Gr}}_{B,x} & & \\
 & \overset{\overline{p}_{B,x}}{\swarrow} & \downarrow {\scriptstyle i_{B,x}} & \overset{\overline{q}_{B,x}}{\searrow} & \\
\mathrm{Gr}_{G,x} & & & & \mathrm{Gr}_{T,x} \\
\downarrow {\scriptstyle i_{G,x}} & & \overline{\mathrm{Gr}}_{B,\mathrm{Ran}} & & \downarrow {\scriptstyle i_{T,x}} \\
 & \underset{\overline{p}_{B,\mathrm{Ran}}}{\swarrow} & & \underset{\overline{q}_{B,\mathrm{Ran}}}{\searrow} & \\
\mathrm{Gr}_{G,\mathrm{Ran}} & & & & \mathrm{Gr}_{T,\mathrm{Ran}},
\end{array}$$

where x is the composition of a with the projection $\mathrm{Gr}_{T,\mathrm{Ran}} \to \mathrm{Ran}$. By factorization, we may assume that x factors through the image of the main diagonal $X \to \mathrm{Ran}$. Base change along this diagram shows that the functor $i^!_{T,x} \circ \mathrm{Jac}^{T}_{!*} : \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to D(\mathrm{Gr}_{T,x})$ is given by:

$$\mathcal{F} \longmapsto \overline{q}_{B,x,*}(i^!_{B,x}({}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}) \overset{!}{\otimes} \overline{p}^!_{B,x}(i^!_{G,x}(\mathcal{F}))).$$

Here, we have denoted by ${}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}$ the !-pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}$ along $\overline{\mathrm{Gr}}_{B,\mathrm{Ran}} \to \mathfrak{L}^+_{\mathrm{Ran}}T\backslash\widetilde{S}^0_{B,\mathrm{Ran}}$.

Applying this to $\mathcal{F} = \psi_{G,\mathrm{Ran}}$ gives

$$i^!_{T,x} \circ \mathrm{Jac}^{T}_{!*}(\psi_{G,\mathrm{Ran}}) \simeq \overline{q}_{B,x,*}(i^!_{B,x}({}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}) \overset{!}{\otimes} \overline{p}^!_{B,x}(\psi_{G,x})),$$

and we want to show that the right hand side is supported only in the 0'th component of $\mathrm{Gr}_{T,x}$.

Let us now further base change along the inclusion of a connected component

$$t^\lambda \longrightarrow \mathrm{Gr}_{T,x}$$

with $\lambda \neq 0$, where we recall that $\mathrm{Gr}_{T,x}$ is (on the level of reduced schemes) the disjoint union of t^λ's, with $\lambda \in \Lambda$.

The fiber of $\overline{\mathrm{Gr}}_{B,x}$ over t^λ is the closure $\overline{S}^\lambda_x$ of the semi-infinite orbit S^λ_x in $\mathrm{Gr}_{G,x}$ indexed by λ. Unwinding the definitions, we have note the !-pullback of ${}_{\mathrm{Gr}_{T,\mathrm{Ran}}}\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}$ to $\overline{S}^\lambda_x$ is equivalent to the translation $t^\lambda\,\mathrm{IC}^{\frac{\infty}{2}}_{B,x}$ of $\mathrm{IC}^{\frac{\infty}{2}}_{B,x} = i^!_{G,x}(\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}})$ by the automorphism induced by multiplying by t^λ. We therefore want to show that

$$H^*_{dR}(\overline{S}^\lambda_x \cap S^{-,0}_x, t^\lambda\,\mathrm{IC}^{\frac{\infty}{2}}_{B,x} \overset{!}{\otimes} \psi_{G,x}) = 0 \tag{6.4.4}$$

for each $\lambda \in \Lambda \backslash 0$.

If $\overline{S}^\lambda_x \cap S^{-,0}_x$ is empty there is nothing to show. Otherwise, it is a closed subscheme of a finite-dimensional scheme $\mathcal{Z}$ called the *Zastava space*. By [Gai21, Prop. 3.8.3], the semi-infinite IC sheaf restricted to the intersection $\overline{S}^\lambda_x \cap S^{-,0}_x$ is equivalent to the perverse intersection cohomology sheaf of $\mathcal{Z}$ restricted to the same intersection, up to a cohomological shift.[25] Hence the vanishing of (6.4.4) is exactly Theorem 3.4.1 of [Ras21]. □

Second proof of vanishing in the principal case. The Hecke structure on $\mathrm{IC}^{\frac{\infty}{2}}_{B,\mathrm{Ran}}$ gives a canonical commutative triangle:

$$\begin{array}{ccc} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{Res}^{\check{G}}_{\check{T}}} & \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}{\otimes} \mathrm{Rep}(\check{T})_{\mathrm{Ran}} \\ & \searrow^{\mathrm{Jac}^T_{!*}} & \downarrow {\scriptstyle \mathrm{Jac}^{T,\mathrm{Hecke}}_{!*}} \\ & & D(\mathrm{Gr}_{T,\mathrm{Ran}}). \end{array}$$

By construction, under geometric Casselman-Shalika, the functor $\mathrm{Res}^{\check{G}}_{\check{T}}$ corresponds to the same named functor $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{T})_{\mathrm{Ran}}$. Hence it suffices to show that taking the fiber of $\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*}$ at a point $x \in X$ coincides with the identity functor

$$\mathrm{id}_{\mathrm{Rep}(\check{T})} : \mathrm{Rep}(\check{T}) \longrightarrow \mathrm{Rep}(\check{T}).$$

Note the functor $\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*,x}$ given by $\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*}$ followed by taking the fiber at x is given by the kernel $\mathrm{IC}^{\frac{\infty}{2}}_{B,x} \in D(\mathrm{Gr}_{G,x})$.

We will carry out the following construction. For each $\lambda \in \Lambda$, we will produce an object $\mathcal{M}^\lambda$ in $\mathrm{Whit}(\mathrm{Gr}_{G,x}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T})$. The collection of $\mathcal{M}^\lambda$'s will satisfy the following:

(1) Each $\mathcal{M}^\lambda$ belongs to the heart of the natural t-structure of the category

$$\mathrm{Whit}(\mathrm{Gr}_{G,x}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}) \simeq \mathrm{Rep}(\check{T}).$$

[25] We remark that given Theorem 5.3.8.1, the proof of Proposition 3.8.3 in [Gai21] reduces to a relative version of [Gai18, Prop. 3.6.5(a)], which compares the IC-sheaf on $\overline{\mathrm{Bun}}_N$ and the Zastava space.

(2) For each λ, there is a surjective map

$$\mathcal{M}^\lambda \longrightarrow k^\lambda,$$

where k^λ is the 1-dimensional representation of $\check{T}$ given by λ.

(3) The functor Φ^λ given by $\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*,x}$ followed by taking the λ-component in $\mathrm{Rep}(\check{T})$ is corepresented by $\mathcal{M}^\lambda$.

(4) We have that

$$\mathrm{Hom}_{\mathrm{Rep}(\check{T})}(\mathcal{M}^\lambda, \mathcal{M}^\mu) = k$$

if $\mu = \lambda$, and is 0 otherwise.

Assuming the existence of the $\mathcal{M}^\lambda$'s, let us finish the proof. First, (1) and (4) together imply that $\mathcal{M}^\lambda$ is irreducible, and hence that the map $\mathcal{M}^\lambda \to k^\lambda$ from (2) is an isomorphism. By (3), for every

$$V \in \mathrm{Rep}(\check{T}),$$

we therefore have

$$\Phi^\lambda(V) = \mathrm{Hom}_{\mathrm{Rep}(\check{T})}(\mathcal{M}^\lambda, V) \simeq \mathrm{Hom}_{\mathrm{Rep}(\check{T})}(k^\lambda, V) = V(\lambda),$$

where $V(\lambda)$ is the λ-component of the $\check{T}$-representation V. But

$$\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*,x} = \bigoplus_{\lambda \in \Lambda} \Phi^\lambda,$$

from which we conclude that $\mathrm{Jac}^{T,\mathrm{Hecke}}_{!*,x}$ is the identity.

Let us now define the $\mathcal{M}^\lambda$'s. First, define

$$\mathcal{M}^0 := \mathrm{Av}_*^{\mathfrak{L}_x N^-,\psi}(\mathrm{gr}\,\mathrm{IC}^{\frac{\infty}{2}}_{B,x}).$$

Here, $\mathrm{gr}\,\mathrm{IC}^{\frac{\infty}{2}}_{B,x}$ is the graded Hecke object corresponding to $\mathrm{IC}^{\frac{\infty}{2}}_{B,x}$. That is,

$$\mathrm{gr}\,\mathrm{IC}^{\frac{\infty}{2}}_{B,x} \in D(\mathrm{Gr}_{G,x}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T})$$

is the image of $\mathrm{IC}^{\frac{\infty}{2}}_{B,x}$ under the functor

$$\mathrm{Hecke}_{\check{M},\check{G}}(\mathrm{SI}_{P,x}) = \mathrm{SI}_{P,x} \underset{\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}) \to \mathrm{SI}_{P,x} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}) \to D(\mathrm{Gr}_{G,x}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}).$$

Here, the first functor is right adjoint to the projection:

$$\mathrm{SI}_{P,x} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}) \to \mathrm{SI}_{P,x} \underset{\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{T}).$$

The second functor is induced by the forgetful functor $\mathrm{SI}_{P,x} \to D(\mathrm{Gr}_{G,x})$.

For an arbitrary $\lambda \in \Lambda$, let $\mathcal{M}^\lambda := \mathcal{M}^0 \otimes k^\lambda$.

The properties (1) – (4) can be verified following [GL19]. To direct the reader to the relevant arguments: Property (1) is proved in the same way as Corollary 25.2.3 in *loc.cit.*, Property (2) is Corollary 25.4.4, Property (3) is the content of Section 26, and Property (4) is Theorem 25.5.2. ☐

Appendix A. Colimit description of intersection cohomology sheaf

The goal of this Appendix is to prove lemma 4.4.8.1 and lemma 4.5.7.1. To do so, we will realize (the fibers of) $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ as a suitable colimit, following [Gai18].

A.1. **Proof of Lemma 4.4.8.1.**

A.1.1. Let $i_x : x \to X \to \mathrm{Ran}$ be a k-point of X. Let $\mathrm{IC}^{\frac{\infty}{2}}_{P,x} := i_x^!(\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}})$. We have

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,x} = \mathrm{Ind}^{\mathrm{Hecke}_{\check{M},\check{G}}}_{\mathrm{DrPl}_{\check{M},\check{G}}}(\delta_{\mathrm{Gr}_{G,x}}),$$

where we now consider $D(\mathfrak{L}^+_x M\backslash \mathrm{Gr}_{G,x})$ as acted on by $\mathrm{Rep}(\check{M})\otimes\mathrm{Rep}(\check{G})$,[26] and with $\delta_{\mathrm{Gr}_{G,x}}$ defining an object of $\mathrm{DrPl}(D(\mathfrak{L}^+_x M\backslash \mathrm{Gr}_{G,x}))$. That is:

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,x} \simeq \mathrm{Fun}(\check{N}_P\backslash\check{G}) \underset{\mathrm{Fun}(\check{N}_P\backslash\check{G})}{\otimes} \delta_{\mathrm{Gr}_{G,x}}.$$

A.1.2. Let $\Lambda^+_M \subseteq \Lambda$ be the set of dominant cocharacters that are orthogonal to the roots of M. That is, $\lambda \in \Lambda$ lies in Λ^+_M if and only if it is dominant and saitisfies $\langle \alpha_i, \lambda\rangle = 0$ for all $\alpha_i \in \mathcal{J}_M$.

Note that such a λ defines a character of $\check{M}$, and so the corresponding highest weight representation V^λ_M of M is 1-dimensional. In this case, we write $e^\lambda := V^\lambda_M$. We continue to write V^λ for the highest weight representation of $\check{G}$ corresponding to λ.

A.1.3. We consider Λ^+_M as a poset via the relation $\lambda_1 \leqslant \lambda_2 \Leftrightarrow \lambda_2 - \lambda_1 \in \Lambda^+_M$. We note that this poset is filtered due to the assumption that G is semisimple and simply connected.

A.1.4. Let $\mathcal{C}$ be a category acted on by $\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G})$, and let $c \in \mathrm{DrPl}(\mathcal{C})$ be an object of c equipped with a Drinfeld-Plücker structure. We remind (cf. §4.1.5) that these correspond in particular to a coherent family of maps

$$e^\lambda \star c \to c \star V^\lambda, \quad \lambda \in \Lambda^+_M. \tag{A.1.1}$$

In this case, we get a natural functor $\Lambda^+_M \to \mathcal{C}$ given by

$$\lambda \mapsto e^{-\lambda} \star c \star V^\lambda,$$

and for $\lambda_2 = \lambda_1 + \lambda$, the corresponding transition map is:

$$e^{-\lambda_1} \star c \star V^{\lambda_1} \to e^{-\lambda_1} \star e^{-\lambda} \star c \star V^\lambda \star V^{\lambda_1} \to e^{-\lambda_2} \star c \star V^{\lambda_2}.$$

Here, the first map is given by $c \to e^{-\lambda} \star c \star V^\lambda$ induced by (A.1.1), and the last map is induced by the Plücker map $V^\lambda \otimes V^{\lambda_1} \to V^{\lambda+\lambda_1}$

As such, we may form the corresponding colimit:

$$\underset{\lambda\in\Lambda^+_M}{\mathrm{colim}}\, e^{-\lambda} \star c \star V^\lambda.$$

[26] We remind that the action of $\mathrm{Rep}(\check{M})$ is always twisted by the functor (4.3.1).

A.1.5. We have the following generalization of [Gai18, Prop. 6.2.4] to the parabolic setting. The proof follows that of *loc.cit.*

Lemma A.1.5.1. *The functor*

$$\mathrm{DrPl}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{Ind}_{\mathrm{DrPl}_{\check{M},\check{G}}}^{\mathrm{Hecke}_{\check{M},\check{G}}}} \mathrm{Hecke}_{\check{M},\check{G}}(\mathcal{C}) \xrightarrow{\mathrm{oblv}} \mathcal{C}$$

coincides with the functor

$$c \mapsto \underset{\lambda \in \Lambda_M^+}{\mathrm{colim}}\, e^{-\lambda} \star c \star V^{\lambda}.$$

Proof. It suffices to consider the universal case where $\mathcal{C} = \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G})$ and $c = \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}})$.

We have a natural map of $\check{M} \times \check{G}$-representations:

$$\underset{\lambda \in \Lambda_M^+}{\mathrm{colim}}\, e^{-\lambda} \otimes \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \otimes V^{\lambda} \to \mathrm{Fun}(\check{G})$$

induced by the algebra map $\mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \to \mathrm{Fun}(\check{G})$ and then multiplying by the function $g \mapsto \langle \phi, g \cdot v \rangle$. Here, $\phi \in (V^{\lambda})^*$ is the lowest weight vector $e^{-\lambda} \to (V^{\lambda})^*$ and $v \in V^{\lambda}$.

We need to verify that this map is an isomorphism. Since Λ_M^+ is filtered, the colimit lies in the heart of the t-structure of $\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G})$. As such, it suffices to show that the map is an isomorphism after applying the functor $\mathrm{Hom}_G(V^{\mu}, -)$ for each dominant coweight μ.

On the one hand, we have $\mathrm{Hom}_{\check{G}}(V^{\mu}, \mathrm{Fun}(\check{G})) \simeq (V^{\mu})^*$.

On the other hand, suppose $\lambda \in \Lambda_M^+$ is sufficiently large compared to μ. That is, $\lambda + \nu$ is dominant for all M-dominant coweights ν appearing as highest weights in $(V^{\mu})^*$ considered as a $\check{M}$-representation. For such an M-dominant coweight, we write $V_{\check{M}}^{\nu}$ for the corresponding highest weight representation of $\check{M}$.

In this case, we have:

$$\mathrm{Hom}_{\check{G}}(V^{\mu}, e^{-\lambda} \otimes \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \otimes V^{\lambda}) \simeq e^{-\lambda} \otimes \mathrm{Hom}_{\check{G}}(k, \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \otimes (V^{\mu})^* \otimes V^{\lambda}).$$

By Lemma A.1.5.2 below, we may further rewrite this as:

$$\begin{aligned} &= \bigoplus_{\nu} e^{-\lambda} \otimes \mathrm{Hom}_{\check{G}}(k, \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \otimes V^{\lambda+\nu} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^{\nu}, (V^{\mu})^*)) \\ &\simeq \bigoplus_{\nu} e^{-\lambda} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^{\nu}, (V^{\mu})^*) \otimes \mathrm{Hom}_{\check{G}}((V^{\lambda+\nu})^*, \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}})) \\ &\simeq \bigoplus_{\nu} e^{-\lambda} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^{\nu}, (V^{\mu})^*) \otimes V_{\check{M}}^{\lambda+\nu} \\ &\simeq \bigoplus_{\nu} \mathrm{Hom}_{\check{M}}(V_{\check{M}}^{\nu}, (V^{\mu})^*) \otimes V_{\check{M}}^{\nu} \simeq (V^{\mu})^*. \end{aligned}$$

By construction, the induced map

$$(V^{\mu})^* \simeq \mathrm{Hom}_{\check{G}}(V^{\mu}, e^{-\lambda} \otimes \mathrm{Fun}(\overline{\check{N}_P \backslash \check{G}}) \otimes V^{\lambda}) \to \mathrm{Hom}_{\check{G}}(V^{\mu}, \mathrm{Fun}(\check{G})) \simeq (V^{\mu})^*$$

is the identity, as required. □

Lemma A.1.5.2. *Let $\lambda \in \Lambda_M^+$ and let V be a $\check{G}$-representation. Suppose that for all M-dominant coweights ν that appear as highest weights in $\mathrm{Res}_{\check{M}}^{\check{G}}(V)$, the coweight $\lambda + \nu$ is dominant (for $\check{G}$). Then we have a canonical decomposition*

$$V \otimes V^{\lambda} \simeq \bigoplus_{\nu} V^{\lambda+\nu} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^{\nu}, V)$$

as $\check{G}$-representations. Here, the direct sum is over all M-dominant coweights ν.

Proof. Denote by
$$\mathrm{Res}^{\check{G}}_{\check{P}} : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{P})$$
the restriction functor, and denote by
$$\mathrm{coInd}^{\check{G}}_{\check{P}} : \mathrm{Rep}(\check{P}) \to \mathrm{Rep}(\check{G})$$
its right adjoint given by coinduction. Then we have:
$$\begin{aligned} V \otimes V^{\lambda} &\simeq V \otimes \mathrm{coInd}^{\check{G}}_{\check{P}}(e^{\lambda}) \simeq \mathrm{coInd}^{\check{G}}_{\check{P}}(\mathrm{Res}^{\check{G}}_{\check{P}}(V) \otimes e^{\lambda}) \\ &\simeq \bigoplus_{\nu} \mathrm{coInd}^{\check{G}}_{\check{P}}(V^{\nu}_{\check{M}} \otimes e^{\lambda}) \otimes \mathrm{Hom}_{\check{M}}(V^{\nu}_{\check{M}}, V) \\ &\simeq \bigoplus_{\nu} \mathrm{coInd}^{\check{G}}_{\check{P}}(V^{\lambda+\nu}_{\check{M}}) \otimes \mathrm{Hom}_{\check{M}}(V^{\nu}_{\check{M}}, V) \\ &\simeq \bigoplus_{\nu} V^{\lambda+\nu} \otimes \mathrm{Hom}_{\check{M}}(V^{\nu}_{\check{M}}, V). \end{aligned}$$
□

A.1.6. We proceed with the proof of Lemma 4.4.8.1. Recall the semi-infinite IC sheaf:
$$\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}} \in D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$
First, we need to verify that $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is equivariant for the group $\mathfrak{L}_{\mathrm{Ran}} N_P$. That is, we need to verify that $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ lies in the subcategory
$$\mathrm{SI}_{M,\mathrm{Ran}} = D(\mathfrak{L}_{\mathrm{Ran}} N_P \mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}) \subseteq D(\mathfrak{L}^+_{\mathrm{Ran}} M \backslash \mathrm{Gr}_{G,\mathrm{Ran}}).$$
A.1.7. For a category $\mathcal{C}$ and a field-extension $k \subseteq K$, we write $\mathcal{C}_K := \mathcal{C} \underset{\mathrm{Vect}_k}{\otimes} \mathrm{Vect}_K$ for its base change to K.

The following lemma allows us to check equivariance on field-valued points:

Lemma A.1.7.1. *Let $\mathcal{C}, \mathcal{D}$ be $D(\mathrm{Ran})$-module categories and suppose we are given a fully faithful embedding $\mathcal{D} \hookrightarrow \mathcal{C}$. Then an object $c \in \mathcal{C}$ lies in $\mathcal{D}$ if and only if for all finite sets I and field-valued points $j : \mathrm{Spec}(K) \to X^I$, the image of c under the functor:*
$$\mathcal{C} \xrightarrow{-\underset{k}{\otimes} K} \mathcal{C}_K \simeq \mathcal{C}_K \underset{D(\mathrm{Ran})_K}{\otimes} D(\mathrm{Ran})_K \xrightarrow{\mathrm{id} \otimes j^!} \mathcal{C}_K \underset{D(\mathrm{Ran})_K}{\otimes} \mathrm{Vect}_K$$
lies in $\mathcal{D}_K \underset{D(\mathrm{Ran})_K}{\otimes} \mathrm{Vect}_K$.

Proof. It suffices to check that c lies in $\mathcal{D}$ when restricted to $X^I \to \mathrm{Ran}$ for each finite set I. By a standard Cousin complex argument (see e.g. [AGK+20, Lemma 9.2.8]), we may write the dualizing sheaf ω_{X^I} as a colimit of sheaves of the form $j_*(K)$, where $j : \mathrm{Spec}(K) \to X^I_{\mathrm{dR}}$ is a field-valued point. Since ω_{X^I} acts as the identity on $\mathcal{C} \underset{D(\mathrm{Ran})}{\otimes} D(X^I)$, any object in the latter category admits a similar colimit description. □

Lemma A.1.7.2. *The sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is $\mathfrak{L}_{\mathrm{Ran}} N_P$-equivariant.*

Proof.

Step 1. By Lemma A.1.7.1, it suffices to check equivariance when restricting to each field-valued point $\mathrm{Spec}(K) \to X^I$. For ease of notation, we assume that $K = k$. Since $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ is a factorization algebra, we may further assume that the point $\mathrm{Spec}(k) \to X^I$ factors through X. In other words, it suffices to show that $\mathrm{IC}^{\frac{\infty}{2}}_{P,x} \in D(\mathfrak{L}^+_x M \backslash \mathrm{Gr}_{G,x})$ is $\mathfrak{L}_x N_P$-equivariant.

Step 2. Since N_P is unipotent, the loop group $\mathfrak{L}_x N_P$ is an ind-group scheme. That is, we may write $\mathfrak{L}_x N_P$ as a filtered colimit of group schemes $\{N_\alpha\}_{\alpha \in A}$, where A is a filtered category. As such, it suffices to establish equivariance against each N_α.

The subset $\Lambda^+_{M,\alpha} \subseteq \Lambda^+_M$ of those λ such that $\mathrm{Ad}_{t^\lambda}(N_\alpha) \subseteq N(O_x)$ is cofinal. For such a λ and $\mathcal{F} \in D(\mathfrak{L}^+_x N_P \mathfrak{L}^+_x M \backslash \mathrm{Gr}_{G,x})$, note that $e^{-\lambda} \star \mathcal{F}$ is N_α-equivariant. This shows that the colimit

$$\operatorname*{colim}_{\lambda \in \Lambda^+_M} e^{-\lambda} \star \delta_{\mathrm{Gr}_{G,x}} \star V^\lambda$$

is N_α-equivariant. By Lemma A.1.5.1, the above colimit identifies with $\mathrm{IC}^{\frac{\infty}{2}}_{P,x}$. ☐

A.1.8. The above lemma together with the following finish the proof of Lemma 4.4.8.1:

Lemma A.1.8.1. *The sheaf* $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ *is supported on* $\widetilde{S}^0_{P,\mathrm{Ran}}$.

Proof. As in Step 1 of the proof of Lemma A.1.7.2, we may check this after restricting to points $x \in X(k)$. Let $\theta \in \Lambda_{G,P} \backslash \Lambda^{\mathrm{neg}}_{G,P}$ and choose a lift θ' of θ to $\Lambda_{G,P}$. Let $S^{\theta'}_x$ be the corresponding semi-infinite orbit (i.e., the $\mathfrak{L}_x N$-orbit through $t^{\theta'}$). Since $S^{\theta'}_x$ contains $S^\theta_{P,x}$, it suffices to show that the restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{P,x}$ to $S^{\theta'}_x$ is zero. However this follows from the colimit description of $\mathrm{IC}^{\frac{\infty}{2}}_{P,x}$ and the fact that if $S^{\theta'}_x \cap \overline{\mathrm{Gr}^\lambda}_{G,x} \neq \emptyset$, then $\lambda - \theta'$ is a sum of positive simple roots with non-negative integral coefficients. ☐

A.2. Proof of Lemma 4.5.7.1.

A.2.1. Recall the notation of §4.5.6. We need need to show that the sheaf $\mathcal{F}^\theta$ lies in perverse degrees $\geqslant 1 + \langle 2(\rho_G - \rho_M), \theta\rangle$ whenever $\theta \neq 0$.

Since the sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ factorizes over Ran, the sheaf $\mathcal{F} = \bigoplus_{\theta \in \Lambda^{\mathrm{neg}}_{G,P}} \mathcal{F}^\theta$ factorizes over $\mathrm{Conf}_{G,P}$. As such, it suffices to show that $\Delta^!(\mathcal{F}^\theta) \in D(\mathrm{Gr}^{+,\theta}_{M,\mathrm{Conf}} \times_{X^\theta} X)$ lies in perverse degrees $\geqslant 1 + \langle 2(\rho_G - \rho_M), \theta\rangle$, where

$$\Delta : \mathrm{Gr}^{+,\theta}_{M,\mathrm{Conf}} \times_{X^\theta} X \to \mathrm{Gr}^{+,\theta}_{M,\mathrm{Conf}}$$

is induced by the diagonal map:

$$\Delta : X \to X^\theta \quad x \mapsto \theta \cdot x.$$

Lemma A.2.1.1. $\Delta^!(\mathcal{F}^\theta)$ *is ULA over* X.

Proof. By working étale-locally on X, we may assume that $X = \mathbb{A}^1$. In this case, we have:

$$\mathrm{Gr}_{G,X} \simeq \mathrm{Gr}_G \times X, \quad \mathrm{Gr}_{M,X} \simeq \mathrm{Gr}_M \times X, \quad S^\theta_{P,X} \simeq S^\theta_P \times X.$$

Moreover by construction, the restriction of the factorization algebra $\mathcal{O}(\overline{\check{N}_P \backslash \check{G}})$ (resp. $\mathcal{O}(\check{G})$) along

$$(\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \to (\mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} D(X) \simeq \mathrm{Rep}(\check{M}) \otimes \mathrm{Rep}(\check{G}) \otimes D(X)$$

is isomorphic to $\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}}) \boxtimes \omega_X$ (resp. $\mathrm{Fun}(\check{G}) \boxtimes \omega_X$). As such, the restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to $\mathrm{Gr}_{G,X} \simeq \mathrm{Gr}_G \times X$ is isomorphic to $(\mathrm{Fun}(\check{G}) \underset{\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})}{\otimes} \delta_{\mathrm{Gr}_G}) \boxtimes \omega_X$. Since $\mathrm{Gr}^+_{M,X}$ maps isomorphically onto $\mathrm{Gr}^+_{M,\mathrm{Conf}} \underset{X^\theta}{\times} X$, the result follows. ☐

Proof of Lemma 4.5.7.1. Let $x \in X$. Recall that $\mathrm{IC}^{\frac{\infty}{2}}_{P,x}$ denotes the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{P,\mathrm{Ran}}$ to $x \in \mathrm{Ran}$.

Let $i^\theta_x : \mathrm{Gr}^{+,\theta}_{M,x} \to \mathrm{Gr}_{G,x}$ be the natural embedding. By factorization of $\mathcal{F}^\theta$ and Lemma A.2.1.1, it suffices to show that the !-restriction of $\mathcal{F}^\theta$ under

$$\mathrm{Gr}^{+,\theta}_{M,x} \simeq \mathrm{Gr}^{+,\theta}_{M,\mathrm{Conf}} \underset{X^\theta}{\times} \{x\} \to \mathrm{Gr}^{+,\theta}_{M,\mathrm{Conf}}$$

lies in perverse degrees $\geqslant 2 + \langle 2(\rho_G - \rho_M), \theta\rangle$. However, this restriction coincides with $i^{\theta,!}_x(\mathrm{IC}^{\frac{\infty}{2}}_{P,x})$. Recall the isomorphism (see Lemma A.1.5.1):

$$\mathrm{IC}^{\frac{\infty}{2}}_{P,x} \simeq \underset{\lambda \in \Lambda^+_M}{\mathrm{colim}}\, e^{-\lambda} \star \delta_{\mathrm{Gr}_{G,x}} \star V^\lambda.$$

The colimit is filtered, and since the t-structure on $D(\mathrm{Gr}^+_M)$ is compatible with filtered colimits, it suffices to show that $i^{\theta,!}_x(e^{-\lambda} \star \delta_{\mathrm{Gr}_{G,x}} \star V^\lambda)$ lies in perverse cohomological degrees $\geqslant 2 + \langle 2(\rho_G - \rho_M), \theta\rangle$ for any λ.

Let IC^λ be the IC-sheaf on $\overline{\mathrm{Gr}^\lambda_{G,x}}$ corresponding to V^λ. By definition, the assertion exactly amounts to showing that $i^{\lambda+\theta}(\mathrm{IC}^\lambda)$ is in perverse degrees $\geqslant 2$.

Note that when $\theta \neq 0$, the intersection $\mathrm{Gr}^{+,\lambda+\theta}_{M,x} \cap \mathrm{Gr}^\lambda_{G,x}$ is empty. Thus, the result follows from the parity vanishing of IC^λ. ☐

Appendix B. Drinfeld-Plücker structure on delta sheaf

In this section, we construct a natural Drinfeld-Plücker structure on $\delta_{\mathrm{Gr}_G,\mathrm{Ran}}$.

B.0.1. *External convolution.* Recall that the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ comes equipped with an *external convolution* monoidal structure.

For a finite set I, let $\mathrm{Rep}(\check{G})_{X^I} := \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \underset{D(\mathrm{Ran})}{\otimes} D(X^I)$. Let $\star$ denotes the pointwise convolution monoidal structure on $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ defined in §3.2.5. Then external convolution is defined as the composition

$$\mathrm{Rep}(\check{G})_{X^I} \otimes \mathrm{Rep}(\check{G})_{X^J} \to \mathrm{Rep}(\check{G})_{X^{I\sqcup J}} \otimes \mathrm{Rep}(\check{G})_{X^{I\sqcup J}} \xrightarrow{\star} \mathrm{Rep}(\check{G})_{X^{I\sqcup J}}.$$

Here, the first functor comes inserting the unit along $I \to I \sqcup J$ and $J \to I \sqcup J$, cf. the unital structure on $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, see §3.2.3. This defines an external convolution product:

$$\overset{\mathrm{ext}}{\star} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

By construction, applying external convolution to an objects supported over x and y, respectively, the result is supported over $x \sqcup y$.

B.0.2. *Naive geometric Satake.* In addition to the usual (pointwise) convolution structure on the spherical category $\mathrm{Sph}_{G,\mathrm{Ran}}$, this category similarly comes equipped with an external convolution product defined in the same way as for $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$. By construction, the (naive) geometric Satake functor

$$\mathrm{Sat}^{\mathrm{nv}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Sph}_{G,\mathrm{Ran}}$$

is monoidal for both the pointwise and external convolution structures.

B.0.3. We will apply the above setup for $\check{G}$ replaced by $\check{M} \times \check{G}$.

We need to construct an action

$$\mathcal{O}(\overline{\check{N}_P\backslash\check{G}}) \curvearrowright \delta_{\mathrm{Gr}_G,\mathrm{Ran}},$$

where $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})$ is considered as an algebra object of $\mathrm{Rep}(\check{M}\times\check{G})_{\mathrm{Ran}} \simeq \mathrm{Rep}(\check{M}) \underset{D(\mathrm{Ran})}{\otimes} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ with its pointwise convolution product.

By definition of $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})$, cf. §4.2.1, this amounts to a family of compatible maps

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I} \boxtimes \omega_{X^J}) \star \delta_{\mathrm{Gr}_G,X^I} \to \delta_{\mathrm{Gr}_G,X^I} \tag{B.0.1}$$

for each $(I \twoheadrightarrow J) \in \mathrm{TwArr}$.

B.0.4. First suppose that $I = J = *$ is a singleton. Then a map

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}}) \boxtimes \omega_X) \star \delta_{\mathrm{Gr}_G,X} \to \delta_{\mathrm{Gr}_G,X} \tag{B.0.2}$$

amounts to a compatible family of maps

$$V^\lambda_{\check{M},X} \star \delta_{\mathrm{Gr}_G,X} \to \delta_{\mathrm{Gr}_G,X} \star V^\lambda_X \tag{B.0.3}$$

for each dominant coweight λ. As usual, $V^\lambda_{\check{M}}$ denotes the irreducible representation of $\check{M}$ with highest weight λ, and V^λ is the irreducible representation of $\check{G}$ with highest weight λ. The sheaves $V^\lambda_{\check{M},X} \in \mathrm{Sph}_M, V^\lambda_X \in \mathrm{Sph}_G$ denote the corresponding spherical sheaves over X constructed in the usual way (see e.g. [Gai07, App. B]).

We have a canonical such map. Indeed, denote by $\mathrm{IC}^\lambda_M \in D(\mathrm{Gr}_M), \mathrm{IC}^\lambda \in D(\mathrm{Gr}_G)$ the IC-sheaves on $\overline{\mathrm{Gr}^\lambda_M}, \overline{\mathrm{Gr}^\lambda_G}$, respectively. We consider IC^λ_M as a sheaf on Gr_G via the embedding $\mathrm{Gr}_M \to \mathrm{Gr}_G$. From the canonical identification

$$\mathrm{IC}^\lambda[2\langle\rho_G - \rho_M, \lambda\rangle]|_{\mathrm{Gr}^\lambda_M} \simeq \mathrm{IC}^\lambda_M|_{\mathrm{Gr}^\lambda_M},$$

we get a canonical map

$$\mathrm{IC}^\lambda_M \to \mathrm{IC}^\lambda.$$

This gives the desired map (B.0.3).

B.0.5. Note that (B.0.2) gives a map

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}}) \boxtimes \omega_X) \overset{\mathrm{ext}}{\star} \delta_{\mathrm{Gr}_G,X^I} \to \delta_{\mathrm{Gr}_G,X^{I\sqcup *}} \tag{B.0.4}$$

for each finite set I. Indeed, (B.0.4) is obtained from (B.0.2) by applying $\mathrm{ins}_*(\omega_{X^I}\boxtimes -) : D(\mathrm{Gr}_{G,X}) \to D(\mathrm{Gr}_{G,X^{I\sqcup *}})$, where $\mathrm{ins} : X^I \times \mathrm{Gr}_{G,X} \to \mathrm{Gr}_{G,X^{I\sqcup *}}$ inserts an I-tuple of points from which we further restrict the trivialization of the point $(x, \mathcal{P}_G, \alpha) \in \mathrm{Gr}_{G,X}$.

B.0.6. Next, suppose $I = J$. By definition of $\mathrm{Sat}^{\mathrm{nv}}$, we have:

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^I})\simeq\overset{\mathrm{ext}}{\underset{I}{\star}}\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})\boxtimes\omega_X).$$

By iteratively applying (B.0.4), we thus obtain a map

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^I})\overset{\mathrm{ext}}{\star}\delta_{\mathrm{Gr}_G,X^I}\to\delta_{\mathrm{Gr}_G,X^{I\sqcup I}}.$$

!-pulling back along the diagonal $X^I\to X^{I\sqcup I}$, we obtain the map

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^I})\star\delta_{\mathrm{Gr}_G,X^I}\to\delta_{\mathrm{Gr}_G,X^I}. \tag{B.0.5}$$

B.0.7. Finally, let $(I\twoheadrightarrow J)\in\mathrm{TwArr}$ be arbitrary. Let $\Delta_{I\twoheadrightarrow J}:X^J\to X^I$ be the corresponding diagonal embedding. We now get a map:

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^J})\star\delta_{\mathrm{Gr}_G,X^I}\simeq\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\Delta_{I\twoheadrightarrow J,*}(\omega_{X^J}))\star\delta_{\mathrm{Gr}_G,X^I}\to$$
$$\to\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^I})\star\delta_{\mathrm{Gr}_G,X^I}\xrightarrow{(B.0.5)}\delta_{\mathrm{Gr}_G,X^I}.$$

Here, all maps take place in $D(M(O)_{\mathrm{Ran}}\backslash\mathrm{Gr}_{G,\mathrm{Ran}})$. Note that the above factors as:

$$\mathrm{Sat}^{\mathrm{nv}}(\mathrm{Fun}(\overline{\check{N}_P\backslash\check{G}})^{\otimes I}\boxtimes\omega_{X^J})\star\delta_{\mathrm{Gr}_G,X^I}\to\Delta_{I\twoheadrightarrow J*}(\delta_{\mathrm{Gr}_{G,X^J}})\to\delta_{\mathrm{Gr}_{G,X^I}}.$$

B.0.8. As such, we have constructed the necessary family of maps (B.0.1). They naturally come equipped with the higher coherence data required to give an action of $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})$ on $\delta_{\mathrm{Gr}_{G,\mathrm{Ran}}}$.[27]

[27] An alternative way to see this is to use the relative perverse t-structure of Hansen-Scholze [HS23]. Indeed, both $\mathcal{O}(\overline{\check{N}_P\backslash\check{G}})_{X^I}$ and $\delta_{\mathrm{Gr}_{G,X^I}}$ naturally lie in the heart of the relative perverse t-structure. Moreover, pointwise convolution on Sph_{G,X^I} is exact for this t-structure, so the higher compatibilities are automatic.

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