Algebraic Exceptional Set of a Three-Component Curve on Hirzebruch Surfaces

Wei Chen

Abstract. We study the algebraic exceptional set of a three-component curve B with normal crossings on a Hirzebruch surface $\mathbb{F}_e$. If $K_{\mathbb{F}_e} + B$ is big and no component of B is a fiber or the rational curve with negative self-intersection, we prove that the algebraic exceptional set is finite, and in most cases give it an effective bound. We also prove that the algebraic exceptional set coincides with the set of curves that are hyper-bitangent to B.

Contents

1. Introduction

Rational curves on smooth algebraic surfaces are fundamental geometric objects in the study of algebraic surfaces. It is conjectured that there are only finitely many rational curves on a complex smooth projective surface S of general type. In this case, Bogomolov's result [Bog77] implies that there are only finitely many rational curves on S if $c_1^2(S) - c_2(S) > 0$, and later Lu and Miyaoka [LM95] showed that there are only finitely many smooth rational curves on any surface S of general type.

Similarly, rational curves on log surfaces are also of great importance. We are particularly interested in the *algebraic exceptional set*, first introduced by Lang [Lan86], which is defined as follows:

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Definition 1.1. Let (S, B) be a pair, where S is a complex smooth projective surface and B is a reduced curve on S. The algebraic exceptional set associated with this pair is defined as

$$\mathcal{E}(S, B) = \left\{ C \subset S \;\middle|\; \begin{array}{l} C \text{ is an integral rational curve such that} \\ \#\nu_C^{-1}(C \cap B) \leq 2 \end{array} \right\},$$

where $\nu_C \colon C^\nu \to C$ is the normalization map of C. When the surface S is clear from the context, we denote this set simply as $\mathcal{E}(B)$.

In particular, when the pair is of log general type—equivalently, when $K_S + B$ is a big divisor on S—we have the following conjecture:

Conjecture 1. *If (S, B) is a log surface of log general type, then $\mathcal{E}(B)$ is a finite set.*

From an arithmetic point of view, Conjecture 1 can be deduced from Vojta's conjecture in Diophantine geometry [Voj87, Conjecture 3.4.3]; from an analytic-geometric perspective, it is related to the Green–Griffiths–Lang conjecture. For more details, we refer interested readers to [AT20]. It is worth mentioning that McQuillan [McQ98] proved the Green-Griffiths-Lang conjecture for a complex smooth projective surface S of general type with $c_1^2(S) - c_2(S) > 0$.

For the case when S is the projective plane $\mathbb{P}^2$ and B has normal crossing singularities at each intersection point of its irreducible components, it turns out that the fewer irreducible components B has, the more challenging the problem becomes. If B has at most two components, Conjecture 1 is still open, and the only known results are obtained assuming that the curve B is very general, see [Che04], [PR07], [CRY23], [ATY24]; if B has at least four components, Conjecture 1 is easy, see [CT25a, Proposition 4.3.1]; if B has three components, Conjecture 1 is true but requires a non-trivial proof, see [CZ08], [GNSW25]. Recently, Caporaso and Turchet [CT25a] provided a new proof of the three-component case dropping the rationality assumption. Instead of $\mathcal{E}(B)$, they considered the set of integral curves that are *hyper-bitangent* to B:

$$\operatorname{Hyp}(\mathbb{P}^2, B, 2) = \left\{ C \subset \mathbb{P}^2 \;\middle|\; \begin{array}{l} C \text{ is an integral plane curve such that} \\ \#\nu_C^{-1}(C \cap B) \leq 2 \end{array} \right\},$$

and they showed that $\operatorname{Hyp}(\mathbb{P}^2, B, 2) = \mathcal{E}(\mathbb{P}^2, B)$. Moreover, they also proved that $\mathcal{E}(\mathbb{P}^2, B)$ is finite and provided an effective bound for its cardinality.

Similarly, we define $\operatorname{Hyp}(S, B, 2)$ for any pair (S, B) as follows:

Definition 1.2. Let (S, B) be a pair, where S is a complex smooth projective surface and B is a reduced curve on S. The set of hyper-bitangent curves associated with this pair is defined as

$$\mathrm{Hyp}(S, B, 2) = \left\{ C \subset S \;\middle|\; \begin{array}{l} C \text{ is an integral curve such that} \\ \#\nu_C^{-1}(C \cap B) \leq 2 \end{array} \right\}.$$

When the surface S is clear from the context, we denote this set simply as $\mathrm{Hyp}(B, 2)$.

In this paper, we study the case when S is a Hirzebruch surface $\mathbb{F}_e, e \geq 0$ and B is a curve with three irreducible components, in this case Conjecture 1 was open. To prove it, we generalize and apply Caporaso–Turchet method [CT25a] and our results can be summerized in the followig theorem:

Theorem. *Let S be a Hirzebruch surface and B a $3C$-curve on S such that $K_S + B$ is a big divisor. If none of the irreducible components of B is a fiber or the rational curve with negative self-intersection, then*

(1) $\mathcal{E}(B) = \mathrm{Hyp}(B, 2)$*;*
(2) $\mathcal{E}(B)$ *is a finite set.*

This theorem is, to our knowledge, the first extension to a surface other than $\mathbb{P}^2$, of [CZ08]'s finiteness result for the algebraic exceptional set of a three-component curve in $\mathbb{P}^2$. We also prove, under some extra assumptions, that $\mathcal{E}(B) = \emptyset$ if B is general. See Proposition 3.4, Proposition 3.8, and 4.9.

The first part of the theorem is proved in Proposition 3.1, Proposition 3.5, and Proposition 4.3; the second part of the theorem is established in Theorem 3.3, Theorem 3.7, and Theorem 4.6. In fact, we can bound $\mathcal{E}(B)$ effectively except for one special case on $\mathbb{F}_1$, where we obtain the finiteness of $\mathcal{E}(B)$ by applying the method developed by Corvaja and Zannier in [CZ13], and this will be discussed in part (b) of Theorem 4.6. Additionally, in Remark 3.9, we note that on $\mathbb{F}_e, e \geq 2$, the Corvaja–Zannier method in [CZ13] does not apply, in fact our method is independent of [CZ13].

Organization of the Paper. In Section 2, we study the properties of unibranch points in Subsection 2.1, where we introduce Theorem 2.11 and Theorem 2.13, which serve as key tools for our subsequent analysis. We then review basic facts about Hirzebruch surfaces in Subsection 2.2, particularly focusing on a criterion to determine whether an integral divisor is big on Hirzebruch surfaces, as stated in Proposition 2.26.

In Section 3, we investigate $\mathrm{Hyp}(\mathbb{F}_e, B, 2)$ for $e = 0$ and $e \geq 2$. In Section 4, we examine $\mathrm{Hyp}(\mathbb{F}_1, B, 2)$ for the non-minimal Hirzebruch surface $\mathbb{F}_1 \cong \mathrm{Bl}_{pt}\,\mathbb{P}^2$.

Notations. We work over $\mathbb{C}$ unless otherwise stated. Our notations will be consistent with [CT25a].

Throughout the paper, S is an irreducible smooth surface, curves are assumed to be reduced, and curves on a surface S are assumed to be closed in S.

- For a curve B on a smooth projective surface S, we say B is a $3C$**-curve** if B consists of three irreducible components such that B has normal crossing singularities at each intersection point of its components.
- For a $3C$-curve B on a smooth projective surface S, we denote by N the set of intersection points of its components. Write $B = B_1 \cup B_2 \cup B_3$ where the B_i's are the irreducible components of B, then $N = \bigcup_{i<j} B_i \cap B_j$. And by $p_{i,j}$, we mean an intersection point of B_i and B_j.
- We denote the multiplicity of a point p on a curve D as $\mathrm{mult}_p(D)$.
- For an integral curve D, we denote $\nu_D \colon D^\nu \to D$ as the normalization map of D.
- We say a curve D is a rational curve if its normalization has arithmetic genus $p_a(D^\nu) = 0$.
- For a point p on an integral curve D, its δ-invariant, $\delta_p(D)$, is defined as

$$\delta_p(D) = p_a(D) - p_a(D_p^\nu),$$

where D_p^ν is the partial normalization of D at p. It is well-known that

$$\delta_p(D) = \sum_{p'} \frac{m_{p'}(m_{p'} - 1)}{2},$$

where p' runs over all the infinitely near points lying over p, including p itself, and $m_{p'}$ is the multiplicity of p'. See, for instance, [Har77, Chapter V, Example 3.9.3, page 393].
- Let B be a a curve on a smooth projective surface S. We say an integral curve D on S is *hypertangent* to B if

$$\#\nu_D^{-1}(D \cap B) = 1;$$

we say D is *hyper-bitangent* to B if

$$\#\nu_D^{-1}(D \cap B) = 2.$$

- Let B be a curve on a smooth projective surface S, and $\mathcal{C} \in \mathrm{Pic}(S)$ be a divisor class. We define

$$\mathrm{Hyp}_{\mathcal{C}}(B, 2) = \left\{ D \in |\mathcal{C}| \;\middle|\; \begin{array}{l} D \text{ is an integral curve such that} \\ \#\nu_D^{-1}(D \cap B) \leq 2 \end{array} \right\}.$$

Clearly, $\mathrm{Hyp}(B, 2) = \bigcup_{\mathcal{C} \in \mathrm{Pic(S)}} \mathrm{Hyp}_{\mathcal{C}}(B, 2)$. For a point $p \in B$, we define

$$\mathrm{Hyp}_{\mathcal{C}}(B, p) = \left\{ D \in |\mathcal{C}| \;\middle|\; \begin{array}{l} D \text{ is an integral curve which is} \\ \text{hypertangent to } B \text{ at } p \end{array} \right\}$$

and

$$\mathrm{Hyp}^m_{\mathcal{C}}(B, p) = \left\{ D \in \mathrm{Hyp}_{\mathcal{C}}(B, p) \;\middle|\; \mathrm{mult}_p(D) = m \right\}.$$

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2. Preliminaries

2.1. Unibranch Points of Algebraic Curves.

All the results in this subsection hold over an algebraically closed field $\Bbbk$ of arbitrary characteristic.

2.1.1. *Analytic Invariants of Algebraic Curves.*

In this sub-subsection, we firstly collect some facts concerning analytic invariants of algebraic curves. Here by "analytic" we mean the invariant is determined by the completion of the local ring involved. All the results in this sub-subsection are known; however, we include some proofs and references for completeness.

Lemma 2.1. *Let $\varphi\colon C \to D$ be a morphism of curves. If φ induces an isomorphism of completions of local rings on $q \in C$ and $\varphi(q) \in D$, that is, $\varphi^*\colon \hat{\mathcal{O}}_{\varphi(q),D} \to \hat{\mathcal{O}}_{q,C}$ is an isomorphism, then*

$$\operatorname{mult}_q(C) = \operatorname{mult}_{\varphi(q)}(D).$$

Proof. Let $(A, \mathfrak{m})$ be a noetherian local ring, then its $\mathfrak{m}-$adic completion $(\hat{A}, \hat{\mathfrak{m}})$ is also a noetherian local ring which is flat over A, and $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$. By [Sta24, Tag 02M1] we know

$$\operatorname{length}_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}^i) = \operatorname{length}_A(A/\mathfrak{m}^i) \cdot \operatorname{length}_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}) = \operatorname{length}_A(A/\mathfrak{m}^i), \forall i \in \mathbb{N}.$$

Hence, $(A, \mathfrak{m})$ and $(\hat{A}, \hat{\mathfrak{m}})$ have the same Hilbert–Samuel polynomials, therefore, their multiplicities are equal, i.e. $\mu(A) = \mu(\hat{A})$. (For definitions of the Hilbert–Samuel polynomial and the multiplicity of a noetherian local ring, see [Har77, Chapter V, Exercise 3.4, page 394].)

Now, let $(\mathcal{O}_{q,C}, \mathfrak{m}_q)$ be the pair $(A, \mathfrak{m})$ above, we obtain

$$\operatorname{mult}_q(C) = \mu(\mathcal{O}_{q,C}) = \mu(\hat{\mathcal{O}}_{q,C}) = \mu(\hat{\mathcal{O}}_{\varphi(q),D}) = \mu(\mathcal{O}_{\varphi(q),D}) = \operatorname{mult}_{\varphi(q)}(D)$$

as desired. □

Lemma 2.2. *The $\delta-$invariant of a point on a curve is an analytic invariant. Let $\varphi\colon C \to D$ be a morphism of curves. If φ induces an isomorphism of completions of local rings on $q \in C$ and $\varphi(q) \in D$, that is, $\varphi^*\colon \hat{\mathcal{O}}_{\varphi(q),D} \to \hat{\mathcal{O}}_{q,C}$ is an isomorphism, then*

$$\delta_q(C) = \delta_{\varphi(q)}(D).$$

Proof. See [Sta24, Tag 0C3Q] and [Sta24, Tag 0C1R]. □

LEMMA 2.3. *Let q be a point on a curve C, then the number of branches of C at q is an analytic invariant.*

PROOF. See [Sta24, Tag 0C2D] and [Sta24, Tag 0C3Z]. □

LEMMA 2.4. *Let C, D be two curves on a surface S, and let q be a point in $C \cap D$. Then*

$$(C \cdot D)_q = \dim_{\Bbbk} \widehat{\mathcal{O}}_{q,S}/(f_C, f_D),$$

where f_C (respectively f_D) is a local equation defining C (respectively D).

PROOF. By [Har77, Chapter II, Theorem 9.7, page 198], we know

$$\mathcal{O}_{q,S}/(f_C, f_D) \cong \widehat{\mathcal{O}}_{q,S}/(f_C, f_D),$$

as both of them are finite-dimensional $\Bbbk$-vector spaces. Then the lemma follows from the fact that

$$(C \cdot D)_q = \dim_{\Bbbk} \mathcal{O}_{q,S}/(f_C, f_D).$$

□

2.1.2. *Properties of Unibranch Points.*

In this sub-subsection we study unibranch points of algebraic curves.

We firstly introduce the notion of *multiplicity sequence* due to Flenner–Zaidenberg [FZ96]:

DEFINITION 2.5. Let q be a singular unibranch point on a curve C which is contained in a surface S. Let

$$S = V_0 \xleftarrow{\sigma_1} V_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_n} V_n$$

be a minimal resolution of q, C^i be the strict transform of C in V_i and $q_i \in C^i$ be the point lying over q. We also denote $C^0 = C$ and $q_0 = q$.

Let m_i denote the multiplicity of $q_i \in C^i$, then the multiplicity sequence of q is defined to be the following sequence of positive integers:

$$\underline{m}_q = (m_0, m_1, \cdots, m_n).$$

We denote $m := \mathrm{mult}_q(C)$ and by $l(q)$ the smallest integer such that $m_{l(q)} < m$. With these notations, we have $m = m_0 = \cdots = m_{l(q)-1}$ and $m_n = 1$.

Then we have the following lemma by Flenner–Zaidenberg [FZ96, Lemma 1.4] describing the intersection multiplicity of C with another curve which is smooth at q:

LEMMA 2.6. *Let B, C be two curves in $\mathbb{A}^2$, and let $q \in B \cap C$ be a unibranch point for both B and C. If q is a smooth point of B and $\mathrm{mult}_q(C) = m \geq 2$, then either*

$$(B \cdot C)_q = km, \quad l(q) \geq k \geq 1$$

or

$$(B \cdot C)_q = l(q)m + m_{l(q)}.$$

Remark 2.7. By our previous study of analytic invariants, we observe that Lemma 2.6 actually holds over any smooth surface S, as we will see in the proof of Theorem 2.11.

Remark 2.8. In [FZ96, Lemma 1.4], Flenner and Zaidenberg established a stronger result: on $\mathbb{A}^2$, given a singular unibranch point q of a curve C, for any possible intersection multiplicity described in Lemma 2.6, there exists a curve B containing q as a smooth point such that $(B \cdot C)_q$ attains the given intersection multiplicity.

In fact, this works over any surface S: let q be a singular unibranch point of a curve $C \subset S$. Since S is smooth, by [Mil13, Proposition 4.9], there exists an open neighborhood U of q and a regular morphism $\varphi\colon U \to \mathbb{A}^2$ that is étale at q. From Remark 2.7, we know $\varphi(q)$ is a singular unibranch point of $\varphi(C)$ with $\underline{m}_{\varphi(q)} = \underline{m}_q$. By [FZ96, Lemma 1.4], we can find a curve $B \subset \mathbb{A}^2$ such that B contains $\varphi(q)$ as a smooth point and intersects $\varphi(C)$ at $\varphi(q)$ with the prescribed intersection multiplicity. Then $\varphi^{-1}(B)$ contains q as a smooth point and intersects C at q with the same intersection multiplicity.

Corollary 2.9. *Let $q \in C \subset \mathbb{A}^2$ be a singular unibranch point with $\operatorname{mult}_q(C) = m \geq 2$, and let L be the tangent line to C at q. If $(C \cdot L)_q = n < 2m$, then for any curve $B \subset \mathbb{A}^2$ containing q as a smooth point such that B is tangent to C at q, we have*

$$(B \cdot C)_q = n.$$

Proof. By assumption we have $m < n < 2m$. From Lemma 2.6 we see that in the multiplicity sequence $\underline{m}_q$, we have $m_0 = m$ and $m_1 = n - m < m$. Hence, we conclude that $l(q) = 1$, and the result follows immediately. $\square$

Remark 2.10. Let $C := \mathbb{V}(z^{n-m}y^m - x^n) \subset \mathbb{P}^2$ with $m < n < 2m$ and $m \geq 2$. We see that $q = [0 : 0 : 1]$ is a singular unibranch point of C with $\operatorname{mult}_q(C) = m$. Let L be the tangent line to C at q, then $L = \mathbb{V}(y)$ and $(C \cdot L)_q = n$. If B is a curve which contains q as a smooth point and is hypertangent to C at q, then, by Corollary 2.9, we have

$$n = (B \cdot C)_q = (B \cdot C) = n \deg B.$$

This gives $\deg B = 1$, so B must be the tangent line to C at q, meaning $B = L$.

The following theorem will be one of the key tools we will use later:

Theorem 2.11. *Let B, C be two curves on a surface S, and $q \in B \cap C$ be a unibranch point for both B and C. If q is a smooth point of B and $\operatorname{mult}_q(C) = m$, then*

$$\delta_C(q) \geq \frac{(m-1)\left((B \cdot C)_q - 1\right) + \gcd\left((B \cdot C)_q, m\right) - 1}{2} \geq \frac{(m-1)\left((B \cdot C)_q - 1\right)}{2},$$

with equality if $\gcd\left((B \cdot C)_q, m\right) = 1$.

Proof. Since S is smooth, by [Mil13, Proposition 4.9], there exists an open neighborhood U of q and a regular morphism $\varphi\colon U \to \mathbb{A}^2$ that is étale at q.

Let f and g be the defining local equations for $\varphi(B)$ and $\varphi(C)$, respectively. Then, $\varphi^* f$ and $\varphi^* g$ are the local equations defining B and C, respectively. Since φ is étale at q, it induces an isomorphism

$$\varphi^*\colon \hat{\mathcal{O}}_{\varphi(q),\mathbb{A}^2} \to \hat{\mathcal{O}}_{q,S}.$$

Combining Lemma 2.4, we obtain

$$\mathcal{O}_{\varphi(q),\mathbb{A}^2}/(f,g) \cong \hat{\mathcal{O}}_{\varphi(q),\mathbb{A}^2}/(f,g) \cong \hat{\mathcal{O}}_{q,S}/(\varphi^* f, \varphi^* g) \cong \mathcal{O}_{q,S}/(\varphi^* f, \varphi^* g).$$

Hence, we have

$$(B \cdot C)_q = (\varphi(B) \cdot \varphi(C))_{\varphi(q)}.$$

Moreover, by Lemma 2.1 and Lemma 2.3, we know that $\varphi(q)$ is a smooth point of $\varphi(B)$ and a unibranch m-fold point of $\varphi(C)$. By Lemma 2.2, we know that $\delta_C(q) = \delta_{\varphi(C)}(\varphi(q))$. Thus, $\varphi(B)$, $\varphi(C)$, and $\varphi(q)$ satisfy all the assumptions in the statement of the theorem. Therefore, we may assume $S = \mathbb{A}^2$.

Now, in the affine plane, if both B and C are lines, then $m = 1$ and $\delta_C(q) = 0$, so the statement is trivially true.

If $m = 1$, we have $\delta_C(q) = 0$, the statement is also trivially true.

If B intersects C transversally at q, then $(B \cdot C)_q = m$, and we have $\delta_C(q) \geq \frac{m(m-1)}{2} \geq \frac{(m-1)^2}{2}$, as desired.

From now on, we assume $m \geq 2$ and B is tangent to C at q, so $(B \cdot C)_q > m$. Assume $(B \cdot C)_q = km + r$, where $k, r \in \mathbb{N}$, $k \geq 1$, and $m > r \geq 0$.

If $r = 0$, then

$$\delta_C(q) \geq \frac{l(q)m(m-1)}{2} \geq \frac{(B \cdot C)_q \cdot (m-1)}{2},$$

as desired.

If $r > 0$, then by Lemma 2.6, we know that $k = l(q)$.

If $k = 1$ and $r > 0$, let L_q be the tangent line to C at q. By Corollary 2.9, we know that $(B \cdot C)_q = (L_q \cdot C)_q = m + r$, so the result follows from [CT25b, Proposition 5.1.1].

If $k \geq 2$ and $r > 0$, since $l(q) = k \geq 2$, we know that $\mathrm{mult}_{q_1}(C^1) = m$ and $\underline{m}_{q_1} = (m_1, \ldots, m_n)$. Then, we can proceed by induction on k, we obtain

$$\begin{aligned}
\delta_C(q) &= \delta_{C^1}(q_1) + \frac{m(m-1)}{2} \\
&\geq \frac{(m-1)\left((k-1)m + r - 1\right) + \gcd\left((k-1)m + r, m\right) - 1}{2} + \frac{m(m-1)}{2} \\
&= \frac{(m-1)\left(km + r - 1\right) + \gcd(r, m) - 1}{2} \\
&= \frac{(m-1)\left((B \cdot C)_q - 1\right) + \gcd\left((B \cdot C)_q, m\right) - 1}{2}
\end{aligned}$$

as desired. This completes the proof. □

REMARK 2.12. Theorem 2.11 is the generalized local version of the lower bound of δ-invariant in [CT25a, Theorem 2.2.1].

Another key tool we will also need is the following **Strong Triangle Inequality** due to García Barroso–Płoski [GBP15, Theorem 2.8, Corollary 2.9]:

THEOREM 2.13. *Let B, C, D be three curves on a surface S. If they intersect at a point q which is unibranch on all of them, then the smallest two among*

$$\frac{(B \cdot C)_q}{\operatorname{mult}_q(B)\operatorname{mult}_q(C)}, \quad \frac{(B \cdot D)_q}{\operatorname{mult}_q(B)\operatorname{mult}_q(D)}, \quad \frac{(C \cdot D)_q}{\operatorname{mult}_q(C)\operatorname{mult}_q(D)}$$

are equal.

REMARK 2.14. We note that the strong triangle inequality in Theorem 2.13 over $\mathbb{C}$ was first proved by Płoski in [Pło85].

2.2. Basics on Hirzebruch Surfaces.

Recall that the Hirzebruch surface $\mathbb{F}_e$ is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(e))$ for $e \geq 0$. We collect some basic facts about Hirzebruch surfaces here for convenience, with notations consistent with [Har77, Chapter V, Section 2].

THEOREM 2.15. *We have* $\operatorname{Pic}(\mathbb{F}_e) = \mathbb{Z}C_1 \oplus \mathbb{Z}f$, *and*

- $C_1^2 = e$;
- $C_0^2 = -e$, $C_0 \sim C_1 - ef$;
- $f^2 = 0$;
- $(C_0 \cdot f) = 1 = (C_1 \cdot f)$;
- $(C_0 \cdot C_1) = 0$;
- $K_{\mathbb{F}_e} \sim -2C_1 + (e-2)f$.

PROOF. See [Har77, Chapter V, Lemma 2.10, page 373 and Theorem 2.17, page 379]. □

COROLLARY 2.16. *Let $\alpha, \beta \in \mathbb{Z}$. Then the following are equivalent:*

(1) $\alpha C_1 + \beta f$ *is very ample;*
(2) $\alpha C_1 + \beta f$ *is ample;*
(3) $\alpha > 0$ *and* $\beta > 0$.

COROLLARY 2.17. *Let $\alpha, \beta \in \mathbb{Z}$. Then the following are equivalent:*

(1) $|\alpha C_1 + \beta f|$ *contains a smooth irreducible curve;*
(2) $|\alpha C_1 + \beta f|$ *contains an integral curve;*
(3) $\alpha = 0$, $\beta = 1$, *or* $\alpha = 1$, $\beta = -e$, *or* $\alpha > 0$, $\beta \geq 0$.

PROOF. (of the two corollaries) See [Har77, Chapter V, Corollary 2.18, page 380]. □

REMARK 2.18. From Corollary 2.17, we easily see that a divisor $uC_1 + vf = uC_0 + (ue+v)f \in \mathrm{Pic}(\mathbb{F}_e)$ is effective if and only if $u \geq 0$ and $ue+v \geq 0$, because an effective divisor is a formal sum of integral divisors with non-negative coefficients.

LEMMA 2.19. *For an effective divisor $uC_1 + vf \in \mathrm{Pic}(\mathbb{F}_e)$ ($u \geq 0$, $ue + v \geq 0$), if $e \geq 1$, we have*

$$h^0(\mathbb{F}_e, uC_1 + vf) = h^0\left(\mathbb{F}_e, uC_0 + (ue+v)f\right) = \sum_{i=0}^{\min\{u, \lfloor \frac{ue+v}{e} \rfloor\}} (ue + v - ie + 1);$$

if $e = 0$, we have

$$h^0(\mathbb{F}_e, uC_1 + vf) = (u+1)(v+1).$$

PROOF. See [Sas21]. $\square$

REMARK 2.20. When $e \geq 1$, we have $C_0 \sim C_1 - ef$, and by Lemma 2.19, we obtain $h^0(\mathbb{F}_e, C_0) = 1$. Therefore, there exists a unique integral curve in $|C_0|$, which is the unique smooth rational curve on $\mathbb{F}_e, e \geq 1$ with negative self-intersection number. We also denote this curve as C_0.

The following lemma, which follows directly from the adjunction formula, is well known:

LEMMA 2.21. *On $\mathbb{F}_e$, an integral curve $D \in |\alpha C_1 + \beta f|$ has arithmetic genus*

$$p_a(D) = \frac{1}{2}e\alpha(\alpha - 1) + (\alpha - 1)(\beta - 1) = \frac{1}{2}(\alpha - 1)(e\alpha + 2\beta - 2).$$

REMARK 2.22. Let us determine when $p_a(D) = 0$. There are two cases: $\alpha = 1$ and $e\alpha + 2\beta - 2 = 0$.

The first case, $\alpha = 1$, tells us that $|C_1 + \beta f|$ with $\beta \geq 0$ is a linear system whose irreducible members are smooth rational curves.

For the second case, we have $2 = e\alpha + 2\beta$. If $\beta > 0$, then $2 = e\alpha + 2\beta \geq 2\beta \geq 2$, hence the equalities here hold, we must have $\alpha = 0$ and $\beta = 1$, in this case $D \in |f|$ is a fiber. If $\beta = 0$, then $e\alpha = 2$ splits into two subcases: either $e = 1$, $\alpha = 2$ or $e = 2$, $\alpha = 1$. Since the latter case is already included in the $\alpha = 1$ case, the only nontrivial case is $e = 1$, $\alpha = 2$.

When $e = 1$, $\mathbb{F}_1 \cong \mathrm{Bl}_p \mathbb{P}^2$, where p is a point in $\mathbb{P}^2$. Denote the blow-up morphism by $\pi\colon \mathrm{Bl}_p \mathbb{P}^2 \to \mathbb{P}^2$ and the exceptional divisor by E. We have the following correspondences:

(1) $C_1 = \pi^*\mathcal{O}(1)$;
(2) $C_0 = E$;
(3) $f = \pi^*\mathcal{O}(1) - E$.

We see that the only special case (given by $e\alpha + 2\beta - 2 = 0$) for smooth rational curves on $\mathbb{F}_e$ is $\pi^*\mathcal{O}(2) \in \mathrm{Pic}(\mathbb{F}_1)$.

LEMMA 2.23. *For a unibranch point q of an integral curve $C \subset \mathbb{F}_e$, assume $C \in |\alpha C_1 + \beta f|$ and C is not in $|C_0|$ or $|f|$, and let $m = \mathrm{mult}_q(C)$.*

If $e = 0$, then $\alpha, \beta > 0$ and $m \leq \min\{\alpha, \beta\}$.

If $e \geq 1$, then $\alpha > 0$, $\beta \geq 0$, and

(1) $m \leq \min\{\alpha, \beta\}$ if $q \in C_0$;
(2) $m \leq \alpha$ if $q \notin C_0$.

PROOF. If $e = 0$, then $C_1 \sim C_0$, and $|C_0| = |C_1|$ and $|f|$ are two rulings on $\mathbb{F}_0$. Hence, $C \notin |C_0| = |C_1|$ and $C \notin |f|$ imply that $\alpha, \beta > 0$, and there exist unique $C_{1,q} \in |C_1|$ and $f_q \in |f|$ that contain q. Thus, we obtain

$$m \leq \min\{(f_q \cdot C)_q, (C_{1,q} \cdot C)_q\} \leq \min\{(f_q \cdot C), (C_{1,q} \cdot C)\} = \min\{\alpha, \beta\}$$

as desired.

If $e \geq 1$, then $\alpha > 0$ and $\beta \geq 0$ follow immediately from Corollary 2.17. We note that either there exists an integral curve $L \in |C_1|$ such that $q \in L$ and $L \neq C$, or $q \in C_0$.

Let $f_q \in |f|$ be the unique fiber containing q, then

$$m \leq (f_q \cdot C)_q \leq (f_q \cdot C) = (f \cdot (\alpha C_1 + \beta f)) = \alpha.$$

If $q \in L \in |C_1|$, then

$$m \leq (L \cdot C)_q \leq (L \cdot C) = (C_1 \cdot (\alpha C_1 + \beta f)) = e\alpha + \beta.$$

In this case, we get $m \leq \alpha$.

If $q \in C_0$, then

$$m \leq (C_0 \cdot C)_q \leq (C_0 \cdot C) = (C_0 \cdot (\alpha C_1 + \beta f)) = \beta.$$

In this case, we get $m \leq \min\{\alpha, \beta\}$. □

LEMMA 2.24. *For a point q of an integral curve $C \subset \mathbb{F}_1$ where $C \in |dC_1|$ for some $d \geq 2$, we have*

$$\mathrm{mult}_q(C) < d.$$

PROOF. This follows from the fact that $dC_1 = \pi^*\mathcal{O}(d)$, as mentioned in Remark 2.22, and an integral plane curve of degree $d \geq 2$ cannot have a point of multiplicity $\geq d$. □

DEFINITION 2.25. Let X be an irreducible projective variety of dimension m, and let $\mathcal{L} \in \mathrm{Pic}(X)$. The volume of $\mathcal{L}$ is defined as

$$\mathrm{vol}(\mathcal{L}) = \limsup_{n \to \infty} \frac{h^0(X, \mathcal{L}^{\otimes n})}{n^m/m!}.$$

The volume of a Cartier divisor D is defined as $\mathrm{vol}(\mathcal{O}_X(D))$.

It is well-known that $\mathcal{L}$ is big if and only if $\mathrm{vol}(\mathcal{L}) > 0$, see, for instance, [Laz04, Chapter 2, Section 2.2.C, page 148].

PROPOSITION 2.26. *A divisor* $uC_1 + vf \in \mathrm{Pic}(\mathbb{F}_e)$ *is big if and only if* $u > 0$ *and* $ue + v > 0$.

PROOF. First, we note that a big divisor on $\mathbb{F}_e$ must be effective. Indeed, if $uC_1 + vf$ is a big divisor on $\mathbb{F}_e$, then there exist $m \in \mathbb{N}$, an ample divisor A, and an effective divisor F such that $m(uC_1+vf) = m(uC_0+(ue+v)f) \sim A+F$. This implies $mu > 0$ and $m(ue+v) > 0$, so $u > 0$ and $ue + v > 0$. Thus, $uC_1 + vf$ is effective.

For an effective divisor $uC_1 + vf$ ($u \geq 0$, $ue + v \geq 0$), we use Lemma 2.19 to compute its volume $\mathrm{vol}(uC_1 + vf)$.

If $e = 0$, we have

$$h^0(\mathbb{F}_0, n(uC_1 + vf)) = (nu + 1)(nv + 1) = uvn^2 + (u + v)n + 1.$$

Thus,

$$\begin{aligned}\mathrm{vol}(uC_1 + vf) &= \limsup_{n\to\infty} \frac{h^0(\mathbb{F}_0, n(uC_1 + vf))}{n^2/2} \\ &= \limsup_{n\to\infty} \frac{2(uvn^2 + (u + v)n + 1)}{n^2} \\ &= 2uv.\end{aligned}$$

Hence, $uC_1 + vf$ is big if and only if $u > 0$ and $v > 0$.

If $e \geq 1$, we consider the following subcases:

(1) If $u = 0$, then $h^0(\mathbb{F}_e, n(uC_1+vf)) = h^0(\mathbb{F}_e, nvf) = nv+1$. In this subcase, $uC_1 + vf = vf$ is not big.

(2) If $u > 0$ and $v \geq 0$, then

$$\begin{aligned}h^0(\mathbb{F}_e, n(uC_1 + vf)) &= \sum_{i=0}^{nu}(nue + nv - ie + 1) \\ &= ((ue + v)n + 1)(un + 1) - e\sum_{i=0}^{un} i \\ &= u(ue + v)n^2 + (ue + v + u)n + 1 - \frac{eu^2n^2 + eun}{2} \\ &= \left(uv + \frac{eu^2}{2}\right)n^2 + \left(u + v + \frac{eu}{2}\right)n + 1.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{vol}(uC_1 + vf) &= \limsup_{n\to\infty} \frac{\left(uv + \frac{eu^2}{2}\right)n^2 + \left(u + v + \frac{eu}{2}\right)n + 1}{n^2/2} \\ &= 2uv + eu^2 > 0.\end{aligned}$$

Hence, $uC_1 + vf$ is big when $u > 0$ and $v \geq 0$.

(3) If $u > 0$ and $v < 0$, we write $\lfloor \frac{nue+nv}{e} \rfloor = \frac{nue+nv}{e} - \epsilon$, where ϵ is a rational number determined by n, v, e, and $0 \leq \epsilon < 1$. Then

$$\begin{aligned}
h^0(\mathbb{F}_e, n(uC_1 + vf)) &= \sum_{i=0}^{\frac{nue+nv}{e}-\epsilon} (nue + nv - ie + 1) \\
&= ((ue+v)n+1)\left(\frac{(ue+v)n}{e} - \epsilon\right) - e\sum_{i=0}^{\frac{nue+nv}{e}-\epsilon} i \\
&= \frac{(ue+v)^2n^2}{e} + \frac{(ue+v)(1-e\epsilon)n}{e} + \epsilon \\
&\quad - \frac{e}{2}\left(\frac{(ue+v)^2n^2}{e^2} + \frac{(1-2\epsilon)(ue+v)n}{e}\right) - \frac{e}{2}(\epsilon^2 - \epsilon) \\
&= \frac{(ue+v)^2}{2e}n^2 + \frac{(ue+v)(2-e)}{2e}n + \epsilon - \frac{e}{2}(\epsilon^2 - \epsilon).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{vol}(uC_1 + vf) &= \limsup_{n\to\infty} \frac{\frac{(ue+v)^2}{2e}n^2 + \frac{(ue+v)(2-e)}{2e}n + \epsilon - \frac{e}{2}(\epsilon^2-\epsilon)}{n^2/2} \\
&= \frac{(ue+v)^2}{e} \geq 0.
\end{aligned}$$

Since $ue + v \geq 0$, we see that $uC_1 + vf$ is big if and only if $ue + v > 0$ in this subcase.

In summary, an effective divisor $uC_1 + vf$ on $\mathbb{F}_e$ is big if and only if $u > 0$ and $ue + v > 0$ when $e \geq 1$. Combining this with the $e = 0$ case, the proof is complete. □

3. Hyper-bitangent Curves on Minimal Hirzebruch Surfaces

In this section, we investigate $\mathrm{Hyp}(B, 2)$ for a minimal Hirzebruch surface S, that is, $S = \mathbb{F}_e$ for $e = 0$ and $e \geq 2$.

On $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we denote the divisor $uC_1 + vf$ as (u, v) by tradition. Notice that $C_0 = C_1$ in this case. With the polarization $(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$ fixed, $|C_1| = |(1, 0)|$ and $|f| = |(0, 1)|$ give two families of lines. By Proposition 2.26, we see that for a divisor B on $\mathbb{P}^1 \times \mathbb{P}^1$, $K_{\mathbb{P}^1\times\mathbb{P}^1} + B$ is big if and only if $B \geq (3, 3)$.

Then, we have the following:

Proposition 3.1. *Consider $\mathbb{F}_0 \cong \mathbb{P}^1\times\mathbb{P}^1$. Let $B = B_1 \cup B_2 \cup B_3 \in |\alpha C_1 + \beta f|$ be a 3C-curve such that $K_{\mathbb{P}^1\times\mathbb{P}^1} + B$ is big. Assume $B_i \in |(\alpha_i, \beta_i)|$ with $(\alpha_i, \beta_i) \geq (1, 1)$ for all $i = 1, 2, 3$, and $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2 \leq \alpha_3 + \beta_3$. Consider $(d_1, d_2) \geq (1, 1)$. If $\mathrm{Hyp}_{(d_1,d_2)}(B, 2) \neq \emptyset$, then:*

(1) $B_1 \in |(1, 1)|$.

(2) $d_1 = d_2 = 1$.

(3) Furthermore, let $\mathfrak{I} \subseteq \{1,2,3\}$ *be the set of indices such that* $B_i \in |(1,1)|$ *for all* $i \in \mathfrak{I}$. *Then*

$$\mathrm{Hyp}_{(1,1)}(B,2) = \bigcup_{\substack{i\in\mathfrak{I} \\ \{i,j,k\}=\{1,2,3\} \\ p_{i,j}\in B_i\cap B_j \\ p_{i,k}\in B_i\cap B_k}} \mathrm{Hyp}_{(1,1)}(B_j, p_{i,j}) \cap \mathrm{Hyp}_{(1,1)}(B_k, p_{i,k}).$$

In particular,

$$\mathrm{Hyp}(B,2) = \mathrm{Hyp}_{(1,0)}(B,2) \cup \mathrm{Hyp}_{(0,1)}(B,2) \cup \mathrm{Hyp}_{(1,1)}(B,2) = \mathcal{E}(B).$$

Proof. Let $D \in \mathrm{Hyp}_{(d_1,d_2)}(B,2)$. Then, $\#(D\cap B) \leq 2$, and $(D\cdot B_i) \geq 2$ for any $i = 1,2,3$. On the other hand, $B_1 \cap B_2 \cap B_3 = \emptyset$ by assumption. Therefore, we must have $\#(D \cap B) = 2$. This implies that any point in $D \cap B$ is a unibranch point on D.

Claim 3.2. $D \cap B_3 \subseteq N = \bigcup_{i<j} B_i \cap B_j$, and $\#(D \cap B_3) = 1$.

Proof of the Claim. Suppose there exists a point q in $D \cap B_3$ that is not contained in N. Then, D must intersect $B_1 \cup B_2$ in exactly one point. Since $(B_i \cdot B_j) = \alpha_i\beta_j + \alpha_j\beta_i \geq 2$, we know $B_i \cap B_j \neq \emptyset$ for any $i \neq j$. Thus, D intersects $B_1 \cup B_2$ at a unique point $p_{1,2} \in B_1 \cap B_2$, and it must be a unibranch n-fold point of D for some $n \geq 1$. By Lemma 2.23, we obtain

$$n \leq \min\{d_1, d_2\} < d_1 + d_2.$$

As B_1 and B_2 meet transversally at $p_{1,2}$, we see that D is tangent to one of them and transverse to the other. Assume D is transverse to B_i, $i \in \{1,2\}$, then

$$d_1\beta_i + d_2\alpha_i = (D\cdot B_i) = (D \cdot B_i)_{p_{1,2}} = n \leq \min\{d_1,d_2\} < d_1 + d_2,$$

which is a contradiction. Thus, we have proved that $D \cap B_3 \subseteq N$.

Now, if $\#(D\cap B_3) = 2$, then $D \cap B_3 = \{p_{1,3}, p_{2,3}\}$ for some points $p_{1,3} \in B_1 \cap B_3$ and $p_{2,3} \in B_2 \cap B_3$. As B_3 meets B_i transversally at $p_{i,3}$ for any $i = 1,2$, we see that D must be transverse to B_i or B_3 at $p_{i,3}$.

If D is transverse to B_i at $p_{i,3}$ for some $i \in \{1,2\}$, as before, we may assume $p_{i,3}$ is a unibranch n-fold point of D for some $n \geq 1$. Then

$$d_1\beta_i + d_2\alpha_i = (D\cdot B_i) = (D \cdot B_i)_{p_{i,3}} = n \leq \min\{d_1,d_2\} < d_1 + d_2,$$

which is again a contradiction. Hence, D is transverse to B_3 at both $p_{1,3}$ and $p_{2,3}$. Let $m_i \coloneqq \mathrm{mult}_{p_{i,3}}(D)$ for $i = 1,2$, then

$$m_1 + m_2 \leq 2\min\{d_1, d_2\} \leq d_1 + d_2.$$

Therefore, we have

$$d_1\beta_3 + d_2\alpha_3 = (D \cdot B_3) = (D\cdot B_3)_{p_{1,3}} + (D \cdot B_3)_{p_{2,3}} = m_1 + m_2 \leq d_1 + d_2,$$

which is possible only if $\alpha_3 = \beta_3 = 1$. In this case, we must have

$$\#(D \cap B_1) = \#(D \cap B_2) = 1.$$

And by the assumption $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2 \leq \alpha_3 + \beta_3$, we also have

$$B_1, B_2, B_3 \in |(1,1)|.$$

As $D \cap B_1 = \{p_{1,3}\} \subset N$, we see that B_1 satisfies the requirement of the claim if it is in the place of B_3. Thus, we may switch B_1 and B_3, and the claim still holds. This completes the proof of the claim. □

Now, we have proved that $D \cap B_3 = \{p_{i,3}\}$ for some $i \in \{1,2\}$. Notice that D must be tangent to B_3 at $p_{i,3}$ because

$$(D \cdot B_3) = \alpha_3 d_1 + \beta_3 d_2 \geq d_1 + d_2 > \mathrm{mult}_{p_{i,3}}(D).$$

Thus, D must be transverse to B_i at $p_{i,3}$. Then D must meet B_i at a further point, since

$$(D \cdot B_i) \geq d_1 + d_2 > \mathrm{mult}_{p_{i,3}} = (D \cdot B_i)_{p_{i,3}}.$$

As D must meet the other component B_j (where $j \neq i, 3$), we know that there exists a point $p_{1,2} \in B_1 \cap B_2$ such that $D \cap B_j = \{p_{1,2}\}$ and $D \cap B = \{p_{1,2}, p_{i,3}\}$.

Now, B_j and D meet only at $p_{1,2}$. Since $(B_j \cdot D) \geq d_1 + d_2 > \mathrm{mult}_{p_{1,2}}(D)$, we see that D cannot be transverse to B_j at $p_{1,2}$. Hence, it must be transverse to B_i at $p_{1,2}$. By Lemma 2.23, we know

$$\mathrm{mult}_{p_{1,2}}(D) \leq \min\{d_1, d_2\}$$

and

$$\mathrm{mult}_{p_{i,3}}(D) \leq \min\{d_1, d_2\}.$$

Then

$$\begin{aligned} d_1\beta_i + d_2\alpha_i &= (D \cdot B_i) \\ &= (D \cdot B_i)_{p_{1,2}} + (D \cdot B_i)_{p_{i,3}} \\ &= \mathrm{mult}_{p_{1,2}}(D) + \mathrm{mult}_{p_{i,3}}(D) \\ &\leq 2\min\{d_1, d_2\} \\ &\leq d_1 + d_2. \end{aligned}$$

Hence the equalities must hold, we obtain $\alpha_i = \beta_i = 1$ and $d_1 = d_2 = \mathrm{mult}_{p_{1,2}}(D) = \mathrm{mult}_{p_{i,3}}(D)$. By the assumption $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2 \leq \alpha_3 + \beta_3$, we get $B_1 \in |(1,1)|$, which proves part (1) of the proposition.

Without loss of generality, let $i = 1$. Denote $d \coloneqq d_1 = d_2$. Then, $D \in |(d,d)|$ for some $d \geq 1$, and

$$D \in \mathrm{Hyp}_{(d,d)}(B,2) \subseteq \mathrm{Hyp}^d_{(d,d)}(B_2, p_{1,2}) \cap \mathrm{Hyp}^d_{(d,d)}(B_3, p_{1,3}).$$

Thus, both $p_{1,2}$ and $p_{1,3}$ are unibranch d-fold points of D. As discussed above, D meets B_1 transversally at $p_{1,2}$ and $p_{1,3}$. Therefore, D must be hypertangent to B_j at $p_{1,j}$ for all $j = 2, 3$. Then, by Theorem 2.11, for all $j = 2, 3$, we obtain

$$\delta_D(p_{1,j}) \geq \frac{(d-1)(d(\alpha_j + \beta_j) - 1) + d - 1}{2} \geq \frac{(d-1)(2d)}{2} = d(d-1).$$

Then

$$\begin{aligned} 0 \leq p_a(D^\nu) &\leq p_a(D) - \delta_D(p_{1,3}) - \delta_D(p_{2,3}) \\ &\leq (d-1)^2 - d(d-1) - d(d-1) \\ &= -(d+1)(d-1) \\ &\leq 0. \end{aligned}$$

Therefore, the equalities must hold, which implies that $d = 1$ and $p_a(D^\nu) = 0$. This proves part (2) of the proposition.

Following the notation of part (3) of the proposition, if $\mathrm{Hyp}_{(1,1)}(B, 2) \neq \emptyset$, then we must have $1 \in \mathfrak{I}$. If $\mathfrak{I} = \{1\}$, i.e., $\{B_1, B_2, B_3\} \cap |(1,1)| = \{B_1\}$, then we must have $i = 1$. In this case, we have proved

$$\mathrm{Hyp}_{(1,1)}(B, 2) \subseteq \bigcup_{\substack{p_{1,2} \in B_1 \cap B_2 \\ p_{1,3} \in B_1 \cap B_3}} \mathrm{Hyp}_{(1,1)}(B_2, p_{1,2}) \cap \mathrm{Hyp}_{(1,1)}(B_3, p_{1,3}),$$

while the reverse inclusion

$$\mathrm{Hyp}_{(1,1)}(B, 2) \supseteq \bigcup_{\substack{p_{1,2} \in B_1 \cap B_2 \\ p_{1,3} \in B_1 \cap B_3}} \mathrm{Hyp}_{(1,1)}(B_2, p_{1,2}) \cap \mathrm{Hyp}_{(1,1)}(B_3, p_{1,3})$$

is obvious.

If B_2 or B_3 is also contained in $|(1,1)|$, then we may switch it with B_1, and the preceding arguments remain valid. By symmetry, we immediately obtain part (3) of the proposition.

Hence, we conclude that

$$\mathrm{Hyp}(B, 2) = \mathrm{Hyp}_{(1,0)}(B, 2) \cup \mathrm{Hyp}_{(0,1)}(B, 2) \cup \mathrm{Hyp}_{(1,1)}(B, 2) = \mathcal{E}(B).$$

This completes the proof of the theorem. □

Now, we give an effective bound for $\mathcal{E}(B)$ in Proposition 3.1:

THEOREM 3.3. *Consider $\mathbb{F}_0$. Let B be as in Proposition 3.1. Then $\mathcal{E}(B)$ is finite. More precisely,*

$$|\mathcal{E}(B)| \leq \begin{cases} 2|N| & \text{if } \{B_1, B_2, B_3\} \cap |(1,1)| = \emptyset; \\ (\alpha_2 + \beta_2)(\alpha_3 + \beta_3) + 2|N| & \text{if } \{B_1, B_2, B_3\} \cap |(1,1)| = \{B_1\}; \\ 4(\alpha_3 + \beta_3) + 2|N| & \text{if } \{B_1, B_2, B_3\} \cap |(1,1)| = \{B_1, B_2\}; \\ 12 + 2|N| = 24 & \text{if } \{B_1, B_2, B_3\} \subset |(1,1)|. \end{cases}$$

Here $|N| = \sum_{i<j}(B_i \cdot B_j) = \sum_{i<j}(\alpha_i\beta_j + \alpha_j\beta_i)$.

PROOF. If D is an integral curve in $\mathrm{Hyp}_{(1,0)}(B, 2)$, then $(D \cap B_i) \geq 1$ for any $i = 1, 2, 3$, since $(\alpha_i, \beta_i) \geq (1, 1)$ by assumption. Hence, D must contain some point in $N = \bigcup_{i<j} B_i \cap B_j$. Since for any point on $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, there is precisely one integral curve in the linear system $|(1, 0)|$ passing through it, it follows that

$$|\mathrm{Hyp}_{(1,0)}(B, 2)| \leq |N|.$$

By symmetry, we obtain

$$|\mathrm{Hyp}_{(0,1)}(B, 2)| \leq |N|.$$

Recall that by Proposition 3.1, we have

$$\mathcal{E}(B) = \mathrm{Hyp}(B, 2) = \mathrm{Hyp}_{(1,0)}(B, 2) \cup \mathrm{Hyp}_{(0,1)}(B, 2) \cup \mathrm{Hyp}_{(1,1)}(B, 2).$$

If $\{B_1, B_2, B_3\} \cap |(1, 1)| = \emptyset$, then, by Proposition 3.1, we know $\mathrm{Hyp}_{(1,1)}(B, 2) = \emptyset$ and $\mathcal{E}(B) = \mathrm{Hyp}_{(1,0)}(B, 2) \cup \mathrm{Hyp}_{(0,1)}(B, 2)$. In this case, we obtain $|\mathcal{E}(B)| \leq 2|N|$.

If $\mathrm{Hyp}_{(1,1)}(B, 2) \neq \emptyset$, then we must have $B_1 \in |(1, 1)|$.

Let D be a curve in $\mathrm{Hyp}_{(1,1)}(B, 2)$.

If $\{B_1, B_2, B_3\} \cap |(1, 1)| = \{B_1\}$, then, as in the proof of Proposition 3.1, we know $D \cap B_1 = \{p_{1,2}, p_{1,3}\}$ for some points $p_{1,2} \in B_1 \cap B_2$ and $p_{1,3} \in B_1 \cap B_3$. With $p_{1,2}$ and $p_{1,3}$ fixed, there is at most one curve in $\mathrm{Hyp}_{(1,1)}(B, 2)$ that contains both $p_{1,2}$ and $p_{1,3}$. Indeed, assume there are two such curves $D_1, D_2 \in \mathrm{Hyp}_{(1,1)}(B, 2)$, then we have

$$D_1 \cap B_2 = \{p_{1,2}\} = D_2 \cap B_2 \quad \text{and} \quad D_1 \cap B_3 = \{p_{1,3}\} = D_2 \cap B_3.$$

Then

$$(D_1 \cdot B_2)_{p_{1,2}} = (D_1 \cdot B_2) = \alpha_2 + \beta_2 = (D_2 \cdot B_2) = (D_2 \cdot B_2)_{p_{1,2}}.$$

Since $(D_1 \cdot D_2) = 2$ and $D_1 \cap D_2 = \{p_{1,2}, p_{1,3}\}$, we must have

$$(D_1 \cdot D_2)_{p_{1,2}} = 1 = (D_1 \cdot D_2)_{p_{1,3}}.$$

This leads to

$$(D_1 \cdot D_2)_{p_{1,2}} = 1 < \alpha_2 + \beta_2 = (D_1 \cdot B_2)_{p_{1,2}} = (D_2 \cdot B_2)_{p_{1,2}}.$$

Note that $p_{1,2}$ is a smooth point on all three curves D_1, D_2, and B_2. But then, by Theorem 2.13, the smallest two among

$$(D_1 \cdot D_2)_{p_{1,2}}, \quad (D_1 \cdot B_2)_{p_{1,2}}, \quad (D_2 \cdot B_2)_{p_{1,2}}$$

are equal, contradiction.

Since there are $(B_1 \cdot B_2)$ choices for $p_{1,2}$ and $(B_1 \cdot B_3)$ choices for $p_{1,3}$, we conclude that

$$|\mathrm{Hyp}_{(1,1)}(B, 2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3).$$

Summing up, we obtain

$$|\mathcal{E}(B)| \leq (\alpha_2 + \beta_2)(\alpha_3 + \beta_3) + 2|N|$$

in the case $\{B_1, B_2, B_3\} \cap |(1,1)| = \{B_1\}$ as desired.

If $\{B_1, B_2, B_3\} \cap |(1,1)| = \{B_1, B_2\}$, then we may switch B_1 and B_2. By symmetry, we obtain

$$|\mathrm{Hyp}_{(1,1)}(B,2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) + (B_2 \cdot B_1)(B_2 \cdot B_3) = 4(\alpha_3 + \beta_3).$$

Hence

$$|\mathcal{E}(B)| \leq 4(\alpha_3 + \beta_3) + 2|N|.$$

If $\{B_1, B_2, B_3\} \subset |(1,1)|$, then we may switch B_2 and B_3 with B_1. By symmetry, we obtain

$$|\mathrm{Hyp}_{(1,1)}(B,2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) + (B_2 \cdot B_1)(B_2 \cdot B_3) + (B_3 \cdot B_1)(B_3 \cdot B_2) = 12.$$

Note that $|N| = 6$ in this case, hence

$$|\mathcal{E}(B)| \leq 12 + 2|N| = 24.$$

This completes the proof. $\square$

Proposition 3.4. *Consider $\mathbb{F}_0$. Let B be as in Proposition 3.1. If $\alpha_i \geq 3$ and $\beta_i \geq 3$ for all $i \in \{1,2,3\}$ and B is general, then*

$$\mathcal{E}(B) = \emptyset.$$

Proof. We first note that $(1,0)|_{B_i}$ is a base point free linear system on B_i for any $i \in \{1,2,3\}$. For any $p \in B_i$, let $F_p^{(1,0)}$ be the unique curve in $|(1,0)|$ that contains p. If B_i is smooth, then by Bertini's theorem ([**Har77**, Chapter III, Corollary 10.9, page 274]), we know for a general point $p \in B_i$ that $F_p^{(1,0)}$ intersects B_i transversally. By symmetry, we can define $F_p^{(0,1)}$, and the same holds. Define the following set of points

$$T_f(B_i) := \{p \in B_i \mid F_p^{(1,0)} \text{ or } F_p^{(0,1)} \text{ is tangent to } B_i\}.$$

Then $T_f(B_i)$ is a finite set.

We choose B general in the following sense: $B_1 \in |(\alpha_1, \beta_1)|$ is smooth; $B_2 \in |(\alpha_2, \beta_2)|$ is smooth, intersects B_1 transversally, and $B_2 \cap T_f(B_1) = \emptyset$; $B_3 \in |(\alpha_3, \beta_3)|$ is smooth, intersects $B_1 \cup B_2$ transversally, and $B_3 \cap T_f(B_1) = \emptyset = B_3 \cap T_f(B_2)$.

Then, by construction, for any $i < j$, $i, j \in \{1,2,3\}$ and for any point $p_{i,j} \in B_i \cap B_j$, $F_{p_{i,j}}^{(1,0)}$ intersects B_i transversally. We obtain

$$\# F_{p_{i,j}}^{(1,0)} \cap B \geq \# F_{p_{i,j}}^{(1,0)} \cap B_i = (F_{p_{i,j}}^{(1,0)} \cdot B_i) = \beta_i \geq 3.$$

Thus $F_{p_{i,j}}^{(1,0)} \notin \mathrm{Hyp}_{(1,0)}(B,2)$. By the first part of the proof of Theorem 3.3, we know $\mathrm{Hyp}_{(1,0)}(B,2) = \emptyset$. By symmetry, we also obtain $\mathrm{Hyp}_{(0,1)}(B,2) = \emptyset$. Combining with Proposition 3.1, we conclude that

$$\mathcal{E}(B) = \mathrm{Hyp}(B,2) = \mathrm{Hyp}_{(1,0)}(B,2) \cup \mathrm{Hyp}_{(0,1)}(B,2) = \emptyset.$$

$\square$

Now, we consider a $3C$-curve $B = B_1 \cup B_2 \cup B_3$ on the Hirzebruch surface $\mathbb{F}_e$ with $e \geq 2$ such that $K_{\mathbb{F}_e} + B$ is big. Assume $B \in |\alpha C_1 + \beta f|$ and $B_i \in |\alpha_i C_1 + \beta_i f|$ for all $i = 1, 2, 3$.

We note that $\beta \geq -e$. Indeed, by Corollary 2.17, if, say, $\beta_1 < 0$, then $\beta_1 = -e$, $\alpha_1 = 1$ and $B_1 = C_0$. Since B is reduced and contains at most one copy of C_0 as its irreducible component, we get $\beta_2, \beta_3 \geq 0$, and it follows that $\beta \geq -e$.

Since $K_{\mathbb{F}_e} = -2C_1 + (e-2)f$, by Proposition 2.26, we have that $K_{\mathbb{F}_e} + B$ is big if and only if $\alpha > 2$ and $(\alpha - 2)e + (\beta + e - 2) > 0$. The latter condition is equivalent to $\beta + (\alpha - 1)e > 2$.

If $B = B_1 \cup B_2 \cup B_3$ is a $3C$-curve on $\mathbb{F}_e, e \geq 2$, where each $B_i \in |\alpha_i C_1 + \beta_i f|$ satisfies $\alpha_i \geq 1$ and $\beta_i \geq 0$ for $i = 1, 2, 3$, then $\alpha \geq 3$ and $\beta \geq 0$. Consequently, $K_{\mathbb{F}_e} + B$ is big. We have the following:

Proposition 3.5. *Consider $\mathbb{F}_e, e \geq 2$. Let $B = B_1 \cup B_2 \cup B_3 \in |\alpha C_1 + \beta f|$ be a $3C$-curve on $\mathbb{F}_e$, and denote $B_i \in |\alpha_i C_1 + \beta_i f|$ for all $i = 1, 2, 3$. Assume that $K_{\mathbb{F}_e} + B$ is big, and that none of B_1, B_2, B_3 belongs to $|f|$ or is C_0.*

Then

$$\mathrm{Hyp}(B, 2) = \mathrm{Hyp}_{C_1}(B, 2) \cup \mathrm{Hyp}_{C_0}(B, 2) \cup \mathrm{Hyp}_f(B, 2) = \mathcal{E}(B),$$

and $\mathrm{Hyp}_{C_1}(B, 2) \neq \emptyset$ only if $e = 2$ and one of the three components lies in the linear system $|C_1|$.

Proof. By assumption, we have $\alpha_i \geq 1$ and $\beta_i \geq 0$ for all $i = 1, 2, 3$. Let $D \in |d_1 C_1 + d_2 f|$ be an integral curve such that $d_1 > 0$ and $d_2 \geq 0$, which is equivalent to D being neither contained in $|f|$ nor equal to C_0. Then $(D \cdot B_i) \geq e \geq 2$ for all $i \in \{1, 2, 3\}$, hence $D \cap B_i \neq \emptyset$.

Recall that $N = \bigcup_{i<j} B_i \cap B_j$. If $D \in \mathrm{Hyp}(B, 2)$, then $\#(D \cap B) \leq 2$. Since $B_1 \cap B_2 \cap B_3 = \emptyset$, we have $\#(D \cap B) = 2$. This implies that any point in $D \cap B$ is a unibranch point on D.

Claim 3.6. $D \cap B_i \subseteq N$ for all $i \in \{1, 2, 3\}$.

Proof of the Claim. Without loss of generality, we may assume $i = 1$. Suppose there exists a point q in $D \cap B_1$ that is not contained in N, then D must intersect $B_2 \cup B_3$ in exactly one point $p_{2,3} \in B_2 \cap B_3$.

Therefore, D is tangent to one of B_2 and B_3 at $p_{2,3}$ and transverse to the other. Without loss of generality, assume D is transverse to B_2 and $p_{2,3}$ is a unibranch n-fold point of D. Then

$$n = (D \cdot B_2) = ((d_1 C_1 + d_2 f) \cdot (\alpha_2 C_1 + \beta_2 f)) = e\alpha_2 d_1 + \alpha_2 d_2 + \beta_2 d_1 \geq e\alpha_2 d_1 > d_1,$$

which contradicts Lemma 2.23. Thus, we have $D \cap B_1 \subseteq N$ as desired. $\square$

Now, since $D \cap B_i \subset N$ for any $i = 1, 2, 3$, we know there exists j such that $\#(D \cap B_j) = 2$ and $\#(D \cap B_i) = 1$ for any $i \neq j$. Without loss of generality, let

$j = 1$, i.e., $\#(D \cap B_1) = 2$. Then

$$D \cap B = D \cap B_1 = \{p_{1,2}, p_{1,3}\},$$

where $p_{1,2} \in B_1 \cap B_2$ and $p_{1,3} \in B_1 \cap B_3$.

If D is transverse to B_i at $p_{1,i}$ for some $i \in \{2, 3\}$, as before, assume $p_{1,i}$ is a unibranch n-fold point of D for some $n \geq 1$. Then

$$\begin{aligned} n = (D \cdot B_i)_{p_{1,i}} = (D \cdot B_i) = ((d_1 C_1 + d_2 f) \cdot (\alpha_i C_1 + \beta_i f)) \\ = e\alpha_i d_1 + \alpha_i d_2 + \beta_i d_1 \geq e d_1 > d_1, \end{aligned}$$

which contradicts Lemma 2.23.

Hence, D is transverse to B_1 at both $p_{1,2}$ and $p_{1,3}$. Let $m_i := \mathrm{mult}_{p_{1,i}}(D)$ for $i = 2, 3$. Then

$$\begin{aligned} 2d_1 \geq m_2 + m_3 = (D \cdot B_1)_{p_{1,2}} + (D \cdot B_1)_{p_{1,3}} = (D \cdot B_1) \\ = e\alpha_1 d_1 + \alpha_1 d_2 + \beta_1 d_1 \geq e d_1 \geq 2d_1. \end{aligned}$$

Thus, all the equalities must hold, which implies

$$e = 2, \quad \alpha_1 = 1, \quad \beta_1 = 0, \quad d_2 = 0, \quad m_2 = m_3 = d_1.$$

From this, we conclude that

$$\mathrm{Hyp}_{d_1 C_1 + d_2 f}(B, 2) = \emptyset$$

if

(1) $e \geq 3$, $d_1 > 0$, $d_2 \geq 0$;
(2) $e = 2$, $d_1 > 0$, $d_2 > 0$.

We are left with the case $e = 2$, $\alpha_1 = 1$, $\beta_1 = 0$, $d_2 = 0$, and $m_2 = m_3 = d_1$. Now, $D \in |d_1 C_1|$. By Theorem 2.11, for any $i = 2, 3$, we have

$$\delta_D(p_{1,i}) \geq \frac{(d_1 - 1)\left((D \cdot B_i) - 1\right)}{2} = \frac{(d_1 - 1)(2\alpha_i d_1 + \beta_i d_1 - 1)}{2}.$$

Hence, we obtain

$$\begin{aligned} 0 \leq p_a(D^\nu) &\leq p_a(D) - \delta_D(p_{1,2}) - \delta_D(p_{1,3}) \\ &\leq \frac{1}{2}(d_1 - 1)(2d_1 - 2) - \frac{1}{2}(d_1 - 1)(2\alpha_2 d_1 + \beta_2 d_1 - 1) \\ &\quad - \frac{1}{2}(d_1 - 1)(2\alpha_3 d_1 + \beta_3 d_1 - 1) \\ &= \frac{1}{2}(d_1 - 1)d_1(2 - 2\alpha_2 - 2\alpha_3 - \beta_2 - \beta_3) \\ &= -(d_1 - 1)d_1 \\ &\leq 0. \end{aligned}$$

Thus, all the equalities here must hold, which implies $d_1 = 1$ and $p_a(D^\nu) = 0$. Therefore, when $e = 2$, for $d_1 > 0$ and $d_2 \geq 0$, the set $\mathrm{Hyp}_{d_1C_1+d_2f}(B,2)$ is non-empty only if $d_1 = 1$, $d_2 = 0$, and $B_1 \in |C_1|$. This completes the proof. $\square$

We obtain an effective bound for $\mathcal{E}(B)$ in Proposition 3.5:

THEOREM 3.7. *Consider $\mathbb{F}_e, e \geq 2$. Let B be as in Proposition 3.5. Then $\mathcal{E}(B)$ is finite. More precisely:*

If $e \geq 3$, then

$$|\mathcal{E}(B)| \leq 1 + |N|.$$

If $e = 2$, assume $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2 \leq \alpha_3 + \beta_3$, then

$$|\mathcal{E}(B)| \leq \begin{cases} 1 + |N| & \text{if } \{B_1, B_2, B_3\} \cap |C_1| = \emptyset; \\ 1 + (2\alpha_2 + \beta_2)(2\alpha_3 + \beta_3) + |N| & \text{if } \{B_1, B_2, B_3\} \cap |C_1| = \{B_1\}; \\ 1 + 4(2\alpha_3 + \beta_3) + |N| & \text{if } \{B_1, B_2, B_3\} \cap |C_1| = \{B_1, B_2\}; \\ 1 + 12 + |N| = 19 & \text{if } \{B_1, B_2, B_3\} \subset |C_1|. \end{cases}$$

Here $|N| = \sum_{i<j}(B_i \cdot B_j) = \sum_{i<j}(e\alpha_i\alpha_j + \alpha_i\beta_j + \alpha_j\beta_i)$.

PROOF. First, by Remark 2.20, we have $|\mathrm{Hyp}_{C_0}(B,2)| \leq 1$.

Next, by the same argument as in the proof of Theorem 3.3 for the $\mathbb{F}_0$ case, we see that any integral curve $D \in \mathrm{Hyp}_f(B,2)$ contains some point in N. Therefore, we obtain

$$|\mathrm{Hyp}_f(B,2)| \leq \sum_{i<j}(B_i \cdot B_j) = |N|.$$

Recall that by Proposition 3.5, we have

$$\mathcal{E}(B) = \mathrm{Hyp}(B,2) = \mathrm{Hyp}_{C_0}(B,2) \cup \mathrm{Hyp}_{C_1}(B,2) \cup \mathrm{Hyp}_f(B,2).$$

If $e \geq 3$, then $\mathrm{Hyp}_{C_1}(B,2) = \emptyset$ and $\mathcal{E}(B) = \mathrm{Hyp}_{C_0}(B,2) \cup \mathrm{Hyp}_f(B,2)$. Hence

$$|\mathcal{E}(B)| \leq 1 + \sum_{i<j}(B_i \cdot B_j) = 1 + |N|.$$

If $e = 2$ and $\{B_1, B_2, B_3\} \cap |C_1| = \emptyset$, then by Proposition 3.5, we know $\mathrm{Hyp}_{C_1}(B,2) = \emptyset$ and $\mathcal{E}(B) = \mathrm{Hyp}_{C_0}(B,2) \cup \mathrm{Hyp}_f(B,2)$. Hence

$$|\mathcal{E}(B)| \leq 1 + |N|.$$

If $e = 2$ and $\mathrm{Hyp}_{C_1}(B,2) \neq \emptyset$, then $\{B_1, B_2, B_3\} \cap |C_1| \neq \emptyset$. By the assumption $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2 \leq \alpha_3 + \beta_3$, we must have $B_1 \in |C_1|$.

Let D be a curve in $\mathrm{Hyp}_{C_1}(B,2)$.

If $\{B_1, B_2, B_3\} \cap |C_1| = \{B_1\}$, then, as in the proof of Proposition 3.5, we know $D \cap B_1 = \{p_{1,2}, p_{1,3}\}$ for some points $p_{1,2} \in B_1 \cap B_2$ and $p_{1,3} \in B_1 \cap B_3$. With $p_{1,2}$ and $p_{1,3}$ fixed, there is at most one curve in $\mathrm{Hyp}_{C_1}(B,2)$ that contains both $p_{1,2}$

and $p_{1,3}$. Indeed, assume there are two such curves $D_1, D_2 \in \mathrm{Hyp}_{C_1}(B, 2)$, then we have

$$D_1 \cap B_2 = \{p_{1,2}\} = D_2 \cap B_2 \quad \text{and} \quad D_1 \cap B_3 = \{p_{1,3}\} = D_2 \cap B_3.$$

Then,

$$(D_1 \cdot B_2)_{p_{1,2}} = (D_1 \cdot B_2) = 2\alpha_2 + \beta_2 = (D_2 \cdot B_2) = (D_2 \cdot B_2)_{p_{1,2}}.$$

Since $(D_1 \cdot D_2) = C_1^2 = 2$ and $D_1 \cap D_2 = \{p_{1,2}, p_{1,3}\}$, we must have

$$(D_1 \cdot D_2)_{p_{1,2}} = 1 = (D_1 \cdot D_2)_{p_{1,3}}.$$

This leads to

$$(D_1 \cdot D_2)_{p_{1,2}} = 1 < 2 \leq 2\alpha_2 + \beta_2 = (D_1 \cdot B_2)_{p_{1,2}} = (D_2 \cdot B_2)_{p_{1,2}}.$$

Note that $p_{1,2}$ is a smooth point on all three curves D_1, D_2, and B_2. But then, by Theorem 2.13, the samllest two among

$$(D_1 \cdot D_2)_{p_{1,2}}, \quad (D_1 \cdot B_2)_{p_{1,2}}, \quad (D_2 \cdot B_2)_{p_{1,2}}$$

are equal, contradiction.

Since there are $(B_1 \cdot B_2)$ choices for $p_{1,2}$ and $(B_1 \cdot B_3)$ choices for $p_{1,3}$, we conclude that

$$|\mathrm{Hyp}_{C_1}(B, 2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) = (2\alpha_2 + \beta_2)(2\alpha_3 + \beta_3)$$

in this case, and hence

$$|\mathcal{E}(B)| \leq 1 + (2\alpha_2 + \beta_2)(2\alpha_3 + \beta_3) + |N|.$$

If $\{B_1, B_2, B_3\} \cap |C_1| = \{B_1, B_2\}$, then we may switch B_1 and B_2. By symmetry, we obtain

$$|\mathrm{Hyp}_{C_1}(B, 2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) + (B_2 \cdot B_1)(B_2 \cdot B_3) = 4(2\alpha_3 + \beta_3),$$

and hence

$$|\mathcal{E}(B)| \leq 1 + 4(2\alpha_3 + \beta_3) + |N|.$$

If $\{B_1, B_2, B_3\} \subset |C_1|$, then we may switch B_2 and B_3 with B_1. By symmetry, we obtain

$$|\mathrm{Hyp}_{C_1}(B, 2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) + (B_2 \cdot B_1)(B_2 \cdot B_3) + (B_3 \cdot B_1)(B_3 \cdot B_2) = 12.$$

Note that $|N| = 6$ in this case, hence we obtain

$$|\mathcal{E}(B)| \leq 1 + 12 + |N| = 1 + 12 + 6 = 19.$$

This completes the proof. □

PROPOSITION 3.8. *Consider $\mathbb{F}_e, e \geq 2$. Let B be as in Proposition 3.5.*

If $\alpha_i \geq 3$ for all $i \in \{1, 2, 3\}$, $\beta_l \geq 3$ for some $l \in \{1, 2, 3\}$, and B is general, then

$$\mathcal{E}(B) = \emptyset.$$

Proof. We first note that $f|_{B_i}$ is a base point free linear system on B_i for any $i \in \{1,2,3\}$. For any $p \in B_i$, let f_p be the unique curve in $|f|$ that contains p. If B_i is smooth, then by Bertini's theorem ([Har77, Chapter III, Corollary 10.9, page 274]), we know for a general point $p \in B_i$ that f_p intersects B_i transversally. Define the following set of points

$$T_f(B_i) := \{p \in B_i \mid f_p \text{ is tangent to } B_i\}.$$

Then $T_f(B_i)$ is a finite set.

We choose B general in the following sense: $B_1 \in |\alpha_1 C_1 + \beta_1 f|$ is smooth; $B_2 \in |\alpha_2 C_1 + \beta_2 f|$ is smooth, intersects B_1 transversally, and $B_2 \cap T_f(B_1) = \emptyset$; $B_3 \in |\alpha_3 C_1 + \beta_3 f|$ is smooth, intersects $B_1 \cup B_2$ transversally, and $B_3 \cap T_f(B_1) = \emptyset = B_3 \cap T_f(B_2)$. Moreover, recall that we have assumed $\beta_l \geq 3$ for some $l \in \{1,2,3\}$, and we also require that B_l intersects C_0 transversally.

Then, by construction, for any $i < j$, $i, j \in \{1,2,3\}$ and for any point $p_{i,j} \in B_i \cap B_j$, $f_{p_{i,j}}$ intersects B_i transversally. We obtain

$$\# f_{p_{i,j}} \cap B \geq \# f_{p_{i,j}} \cap B_i = (f_{p_{i,j}} \cdot B_i) = \alpha_i \geq 3.$$

Thus $f_{p_{i,j}} \notin \mathrm{Hyp}_f(B,2)$. By the first part of the proof of Theorem 3.7, we know $\mathrm{Hyp}_f(B,2) = \emptyset$. Moreover, we have

$$\# C_0 \cap B \geq \# C_0 \cap B_l = (C_0 \cdot B_l) = \beta_l \geq 3.$$

Thus $\mathrm{Hyp}_{C_0}(B,2) = \emptyset$.

Combining with Proposition 3.5, we have

$$\mathcal{E}(B) = \mathrm{Hyp}(B,2) = \mathrm{Hyp}_{C_0}(B,2) \cup \mathrm{Hyp}_f(B,2) = \emptyset.$$

□

Remark 3.9. In [CZ13, Theorem 1, Theorem 2], Corvaja and Zannier bound the degree of a curve on a surface of log-general type which is also a finite affine smooth covering of $\mathbb{G}_m^2$ in terms of the number of intersection points with the boundary divisor and the geometric genus of the curve. We note that, in the case $S = \mathbb{F}_e, e \geq 2$, if, in addition to the assumption in Proposition 3.5, none of the three components is (very) ample, then the Corvaja–Zannier method does not apply. Indeed, in this case the rational curve with negative self-intersection C_0 is contained in $\mathbb{F}_e \setminus B$, hence $\mathbb{F}_e \setminus B$ is not an affine surface. Of course, C_0 cannot be contracted to a point, for it would produce a surface singularity.

4. Hyper-bitangent Curves on $\mathbb{F}_1$

In this section, we investigate $\mathrm{Hyp}(B,2)$ for the non-minimal Hirzebruch surface $S = \mathbb{F}_1 \cong \mathrm{Bl}_{pt}\,\mathbb{P}^2$. It turns out to be a more subtle case than that of the minimal Hirzebruch surfaces.

Now, we consider a $3C$-curve $B = B_1 \cup B_2 \cup B_3$ on $\mathbb{F}_1$ such that $K_{\mathbb{F}_1} + B$ is big. Assume $B \in |\alpha C_1 + \beta f|$ and $B_i \in |\alpha_i C_1 + \beta_i f|$ for $i = 1,2,3$. By Corollary 2.17, if, say, $\beta_1 < 0$, then $\beta_1 = -1$, $\alpha_1 = 1$ and $B_1 = C_0$. As B is reduced, it contains at most one copy of C_0. Hence $\beta_2, \beta_3 \geq 0$. It follows that $\beta \geq -1$.

Since $K_{\mathbb{F}_1} = -2C_1 - f$, by Proposition 2.26, we have that $K_{\mathbb{F}_1} + B$ is big if and only if $\alpha > 2$ and $\alpha + \beta > 3$.

Assume that none of the irreducible components of B belongs to $|f|$ or is C_0; then $\beta \geq 0$. Let us first consider the case $\beta = 0$. Then, $B \in |\alpha C_1|$ with $\alpha \geq 4$, and it does not contain any integral curve of type $uC_1 + vf$ with $u \geq 0$ and $v > 0$ as a component. Hence, $B_i \in |\alpha_i C_1|$ with $\alpha_i \geq 1$ for all $i \in \{1, 2, 3\}$. Consider the blow-up morphism $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$. We have $B = \pi^*\mathcal{O}(\alpha)$ with $\alpha \geq 4$. In this case, the exceptional curve is $E = C_0$ and $B \cap E = \emptyset$ since $(B \cdot E) = \sum_i (\alpha_i C_1 \cdot C_0) = 0$. Thus, for any $D \in \mathrm{Hyp}(B, 2)$, we have $\pi(D) \in \mathrm{Hyp}(\mathbb{P}^2, \pi(B), 2) = \mathcal{E}(\mathbb{P}^2, \pi(B))$, where the last equality holds by [CT25a]. Therefore, D must be a rational curve itself, and we see that the study of $\mathrm{Hyp}(B, 2)$ in this case can be completely reduced to the case on $\mathbb{P}^2$, which has been studied in [CT25a].

Recall that $N = \bigcup_{i<j} B_i \cap B_j$, and $|N| = \sum_{i<j}(B_i \cdot B_j)$ by the definition of a $3C$-curve. For the remaining cases with $\beta > 0$, we have the following proposition and theorems:

Proposition 4.1. *Consider $\mathbb{F}_1$. Let $B = B_1 \cup B_2 \cup B_3 \in |\alpha C_1 + \beta f|$ be a $3C$-curve on $\mathbb{F}_1$, and denote $B_i \in |\alpha_i C_1 + \beta_i f|$ for $i = 1, 2, 3$. Assume that $\beta > 0$, $K_{\mathbb{F}_1} + B$ is big, and that none of B_1, B_2, B_3 is contained in $|f|$ or is C_0. Assume that $\beta_3 \geq 1$ and $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2$.*

Then

$$|\mathrm{Hyp}_{C_1}(B, 2)| \leq \frac{1}{2}|N|(|N| - 1).$$

Proof. By assumption, we have $\alpha_i \geq 1$ and $\beta_i \geq 0$ for all $i = 1, 2, 3$, and $\beta_3 \geq 1$.

Consider the blow-up morphism $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$, we have

$$h^0(\mathbb{F}_1, C_1) = h^0\left(\mathbb{F}_1, \pi^*\mathcal{O}(1)\right) = h^0\left(\mathbb{P}^2, \mathcal{O}(1)\right) = 3.$$

Therefore, through any two points on $\mathbb{F}_1$, there is at most one integral curve in $|C_1|$ which contains both of them. Note that such a curve is a smooth rational curve.

Let $D \in \mathrm{Hyp}_{C_1}(B, 2)$, then $(D \cdot B_i) = \alpha_i + \beta_i \geq 1$ for any $i = 1, 2, 3$, which implies that $D \cap B_i \neq \emptyset$. As $B_1 \cap B_2 \cap B_3 = \emptyset$, we conclude that $\#(D \cap B) = 2$ and at least one of the two points in $D \cap B$ must be contained in N. Let $D \cap B = \{p, q\}$ and $p \in N$. Assume $p \in B_{i_1} \cap B_{i_2}$, $i_1 \neq i_2$; then $q \in B_{i_3}$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. Then D must intersect one of B_{i_1} and B_{i_2} transversally at p. Without loss of generality, let it be B_{i_1}.

Write $\mathrm{Hyp}_{C_1}(B, 2) = H_1 \cup H_2$, where

$$H_1 \coloneqq \left\{D \in \mathrm{Hyp}_{C_1}(B, 2) \;\middle|\; D \cap B \subseteq N\right\}$$

and

$$H_2 \coloneqq \left\{ D \in \mathrm{Hyp}_{C_1}(B,2) \;\middle|\; D \cap B \not\subseteq N \right\}.$$

If $H_2 = \emptyset$, then $D \in H_1$ and $D \cap B = \{p, q\} \subseteq N$. Since there is at most one integral curve in $|C_1|$ which contains both p and q, D must be this curve. We have at most

$$\binom{|N|}{2} = \frac{1}{2}|N|(|N|-1)$$

such curves, therefore the statement of the proposition holds in this case.

It remains to study the case $H_2 \neq \emptyset$.

In this case, for $D \in H_2$, we have $D \cap B_{i_3} = \{q\}$ and $D \cap B_{i_1} = D \cap B_{i_2} = \{p\}$. Note that p is a smooth point on both B_{i_1} and B_{i_2} by the definition of a $3C$-curve. By the assumption that D is transverse to B_{i_1} at p, we have

$$1 = \mathrm{mult}_p(D)\,\mathrm{mult}_p(B_{i_1}) = (B_{i_1} \cdot D)_p = (B_{i_1} \cdot D) = \alpha_{i_1} + \beta_{i_1},$$

which implies that $\alpha_{i_1} = 1$ and $\beta_{i_1} = 0$. Since we have assumed $\beta_3 \geq 1$ and $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2$, we have $i_1 \in \{1, 2\}$ and $B_1 \in |C_1|$.

By contracting C_0, we obtain the blow-up morphism $\pi \colon \mathbb{F}_1 \to \mathbb{P}^2$, and

$$\pi(D), \pi(B_{i_1}) \in |\mathcal{O}(1)|, \quad \pi(B_{i_2}) \in |\mathcal{O}(\alpha_{i_2} + \beta_{i_2})|, \quad \pi(B_{i_3}) \in |\mathcal{O}(\alpha_{i_3} + \beta_{i_3})|.$$

Since $(C_1 \cdot C_0) = 0$, we know $B_{i_1} \cap C_0 = \emptyset = D \cap C_0$ for any $D \in \mathrm{Hyp}_{C_1}(B,2)$. Hence, $\pi(B)$ has normal crossing singularity at $\pi(p)$, and for any $D \in \mathrm{Hyp}_{C_1}(B,2)$, $\pi(D) \subset \mathbb{P}^2$ is a line which does not contain the blow-up point.

Then, we have the following two subcases:

(a) $\{B_1, B_2\} \subset |C_1|$.

In this subcase, $|N| = 1 + 2(\alpha_3 + \beta_3)$. For $D \in H_1$, it cannot coincide with B_1 or B_2, hence one of p and q must be contained in $B_1 \cap B_3$, and the other one must be contained in $B_2 \cap B_3$. Therefore,

$$|H_1| \leq (B_1 \cdot B_3)(B_2 \cdot B_3) = (\alpha_3 + \beta_3)^2.$$

Now consider $D \in H_2$. As discussed before, we have $D \cap B_{i_1} = D \cap B_{i_2} = \{p\}$, $D \cap B_{i_3} = \{q\}$, $q \notin N$, and $B_{i_1} \in |C_1|$ intersects D transversally at p.

If $p \in B_1 \cap B_3$, then $i_1 = 1$ and $D \cap B_3 = \{p\}$. Hence $\pi(D)$ must be the tangent line to $\pi(B_3)$ at $\pi(p)$, and there are at most $(B_1 \cdot B_3) = \alpha_3 + \beta_3$ such lines.

If $p \in B_2 \cap B_3$, then $i_1 = 2$ and $D \cap B_3 = \{p\}$. By symmetry, there are at most $\alpha_3 + \beta_3$ choices for such D.

If $p \in B_1 \cap B_2$, then $\{\pi(p)\} = \pi(B_1) \cap \pi(B_2)$, and $\pi(D)$ is a line containing $\pi(p)$ which is hypertangent to $\pi(B_3)$ at $\pi(q)$. The lines containing $\pi(p)$ give a morphism of degree $\alpha_3 + \beta_3$ from $\pi(B_3)$ to $\mathbb{P}^1$, and those lines which are hypertangent to $\pi(B_3)$ correspond to the points of maximal ramification index $\alpha_3 + \beta_3 - 1$ of this morphism. Denote $b \coloneqq \alpha_3 + \beta_3$;

by the Riemann-Hurwitz theorem for singular curves (see, for instance, [**GL96**, page 2]), we obtain

$$\begin{aligned}\#\text{points of maximal ramification index} &\leq \frac{2p_a\left(\pi(B_3)\right) - 2 - b\left(2p_a(\mathbb{P}^1) - 2\right)}{b-1} \\ &= \frac{(b-1)(b-2) - 2 + 2b}{b-1} = b.\end{aligned}$$

Hence there are at most $b = \alpha_3 + \beta_3$ such lines.

Therefore, we obtain

$$|H_2| \leq 3(\alpha_3 + \beta_3).$$

Summing up, we obtain

$$\begin{aligned}|\mathrm{Hyp}_{C_1}(B,2)| &\leq (\alpha_3+\beta_3)^2 + 3(\alpha_3+\beta_3) \\ &\leq 2(\alpha_3+\beta_3)^2 + (\alpha_3+\beta_3) \\ &= \frac{1}{2}|N|(|N|-1)\end{aligned}$$

in the subcase $\{B_1, B_2\} \subset |C_1|$ as desired.

(b) $\{B_1, B_2\} \cap |C_1| = \{B_1\}$.

In this subcase, we must have $i_1 = 1$. Denote $a \coloneqq \alpha_2 + \beta_2 \geq 2$ and $b \coloneqq \alpha_3 + \beta_3 \geq 2$; then

$$(B_1 \cdot B_2) = a, \quad (B_1 \cdot B_3) = b,$$

and

$$(B_2 \cdot B_3) = \alpha_2\alpha_3 + \alpha_2\beta_3 + \alpha_3\beta_2 = \alpha_2 b + \alpha_3\beta_2 = ab - \beta_2\beta_3.$$

Hence, we obtain $|N| = a + b + ab - \beta_2\beta_3$.

For $D \in H_1$, it cannot contain points in $B_1 \cap B_2$ and $B_1 \cap B_3$ at the same time; otherwise, it coincides with B_1. Therefore, one of p and q must be contained in $B_2 \cap B_3$, and the other one must be contained either in $B_1 \cap B_2$ or $B_1 \cap B_3$. We obtain

$$|H_1| \leq \left[(B_1 \cdot B_2) + (B_1 \cdot B_3)\right](B_2 \cdot B_3) \leq (a+b)(ab - \beta_2\beta_3).$$

For $D \in H_2$, we have $\{p\} = B_1 \cap D = B_{i_2} \cap D$. Since $i_2 \in \{2,3\}$, $\alpha_2 + \beta_2 \geq 2$ and $\alpha_3 + \beta_3 \geq 2$, $\pi(D)$ must be the tangent line to $\pi(B_{i_2})$ at $\pi(p)$. There are at most $(B_1 \cdot B_2) + (B_1 \cdot B_3)$ such lines, hence

$$|H_2| \leq (B_1 \cdot B_2) + (B_1 \cdot B_3) = a + b.$$

Summing up, we obtain

$$|\mathrm{Hyp}_{C_1}(B,2)| = |H_1| + |H_2| \leq (a+b)(ab - \beta_2\beta_3 + 1) = (a+b)(|N| + 1 - a - b).$$

Claim 4.2. $(a+b)(|N|+1-a-b) < \frac{1}{2}|N|(|N|-1)$.

Proof. (of the claim) Denote $c \coloneqq a+b$. Then $c \geq 4$, and

$$|N| = a+b+ab-\beta_2\beta_3 = c+\alpha_2 b+\alpha_3\beta_2 \geq c+b \geq c+2.$$

The statement in the claim is equivalent to

$$|N|^2 - (2c+1)\,|N| + 2c(c-1) > 0.$$

We have

$$\begin{aligned} |N|^2 - (2c+1)\,|N| + 2c(c-1) &= |N|^2 - (2c+1)\,|N| + c(c+1) + c(c-3) \\ &= (|N|-c)\,(|N|-c-1) + c(c-3) > 0 \end{aligned}$$

as $N \geq c+2$ and $c \geq 4$. Hence the statement in the claim holds. $\square$

From the claim, it immediately follows that

$$|\mathrm{Hyp}_{C_1}(B,2)| < \frac{1}{2}|N|(|N|-1)$$

in the subcase $\{B_1, B_2\} \cap |C_1| = \{B_1\}$.

This completes the proof. $\square$

Proposition 4.3. *Consider $\mathbb{F}_1$. Under the same assumptions as in Proposition 4.1, consider $d_1C_1 + d_2 f \in \mathrm{Pic}(\mathbb{F}_1)$ with $d_1 \geq 1$, $d_2 \geq 0$ and $d_1 + d_2 \geq 2$. If $\mathrm{Hyp}_{d_1C_1+d_2f}(B,2) \neq \emptyset$, then the following holds:*

(1) $B_1 \in |C_1|$.

(2) For any $d_1 \geq 2$, let $\mathfrak{I} \subseteq \{1,2\}$ be the set of indices such that $B_i \in |C_1|$ for all $i \in \mathfrak{I}$. Then

$$\mathrm{Hyp}_{d_1C_1}(B,2) = \bigcup_{\substack{i\in\mathfrak{I} \\ \{i,j\}=\{1,2\} \\ p_{1,2}\in B_1\cap B_2 \\ p_{i,3}\in B_i\cap B_3}} \mathrm{Hyp}^{d_1-1}_{d_1C_1}(B_j, p_{1,2}) \cap \mathrm{Hyp}_{d_1C_1}(B_3, p_{i,3}),$$

and for any $d_1 \geq 3$, $\mathrm{Hyp}_{d_1C_1}(B,2) \neq \emptyset$ only if $\{B_1, B_2\} \subset |C_1|$. Moreover, $\left|\bigcup_{d_1\geq 2} \mathrm{Hyp}_{d_1C_1}(B,2)\right|$ is finite and is effectively bounded in terms of α_i's and β_i's.

(3) If $d_2 > 0$, then $d_2 = 1$. Furthermore, let $\mathfrak{I} \subseteq \{1,2\}$ be the set of indices such that $B_i \in |C_1|$ for all $i \in \mathfrak{I}$. Then

$$\mathrm{Hyp}_{d_1C_1+f}(B,2) = \bigcup_{\substack{i\in\mathfrak{I} \\ \{i,j\}=\{1,2\} \\ p_{1,2}\in B_1\cap B_2 \\ p_{i,3}\in B_i\cap B_3}} \mathrm{Hyp}^{d_1}_{d_1C_1+f}(B_j, p_{1,2}) \cap \mathrm{Hyp}_{d_1C_1+f}(B_3, p_{i,3}).$$

Moreover, for any $d_1 \geq 2$, $\mathrm{Hyp}_{d_1C_1+f}(B,2) \neq \emptyset$ only if $\{B_1, B_2\} \subset |C_1|$.

In particular, $\mathrm{Hyp}(B,2) = \mathcal{E}(B)$.

Proof. By assumption, we have $\alpha_i \geq 1$ and $\beta_i \geq 0$ for all $i = 1, 2, 3$, and $\beta_3 \geq 1$.

Let D be a curve in $\mathrm{Hyp}_{d_1 C_1 + d_2 f}(B, 2)$, where $d_1 \geq 1$, $d_2 \geq 0$ and $d_1 + d_2 \geq 2$. Then $D \cap B_i \neq \emptyset$ for all $i = 1, 2, 3$. Since $B_1 \cap B_2 \cap B_3 = \emptyset$, it follows that $\#(D \cap B) = 2$. This implies that any point in $D \cap B$ is a unibranch point on D.

Claim 4.4. $D \cap B_3 \subseteq N$ and $\#(D \cap B_3) = 1$.

Proof of the Claim. Suppose there exists a point q in $D \cap B_3$ that is not contained in N. Then, D must intersect $B_1 \cup B_2$ in exactly one point. Since $(B_i \cdot B_j) = \alpha_i \alpha_j + \alpha_i \beta_j + \alpha_j \beta_i \geq 1$, we know that $B_i \cap B_j \neq \emptyset$ for any $i \neq j$. Thus, D intersects $B_1 \cup B_2$ at one point $p_{1,2} \in B_1 \cap B_2$, which must be a unibranch n-fold point of D for some $n \geq 1$.

Then, D is transverse to one of B_1 and B_2 at $p_{1,2}$. Assume D is transverse to B_i at $p_{1,2}$, $i \in \{1, 2\}$; then

$$n = (D \cdot B_i) = ((d_1 C_1 + d_2 f) \cdot (\alpha_i C_1 + \beta_i f)) = (\alpha_i + \beta_i) d_1 + \alpha_i d_2 \geq d_1 + d_2 \geq d_1.$$

By Lemma 2.23, we see all the equalities here must hold, which gives $\alpha_i = 1$, $\beta_i = 0$ and $d_2 = 0$. But then $D \in |d_1 C_1|$, and $n = d_1 = d_1 + d_2 \geq 2$ contradicts Lemma 2.24. Therefore, we have proved that $D \cap B_3 \subseteq N$.

Now, if $\#(D \cap B_3) = 2$, we have $D \cap B_3 = \{p_{1,3}, p_{2,3}\}$ for some points $p_{1,3} \in B_1 \cap B_3$ and $p_{2,3} \in B_2 \cap B_3$. Then, D is transverse to B_i or B_3 at $p_{i,3}$ for any $i = 1, 2$.

If D is transverse to B_i at $p_{i,3}$ for some $i \in \{1, 2\}$, as before, we may assume $p_{i,3}$ is a unibranch n-fold point of D for some $n \geq 1$. Then

$$n = (D \cdot B_i)_{p_{i,3}} = (D \cdot B_i) = (\alpha_i + \beta_i) d_1 + \alpha_i d_2 \geq d_1 + d_2 \geq d_1.$$

Again, by Lemma 2.23, we see all the equalities here must hold, which implies $\alpha_i = 1$, $\beta_i = 0$ and $d_2 = 0$. But then $D \in |d_1 C_1|$, and $n = d_1 = d_1 + d_2 \geq 2$ contradicts Lemma 2.24.

Hence, D is transverse to B_3 at both $p_{1,3}$ and $p_{2,3}$. Denote $m_i := \mathrm{mult}_{p_{i,3}}(D)$ for any $i = 1, 2$. Then by Lemma 2.23, we know $m_i \leq d_1$ for $i = 1, 2$. We obtain

$$\begin{aligned} 2d_1 \geq m_1 + m_2 &= (D \cdot B_3)_{p_{1,3}} + (D \cdot B_3)_{p_{2,3}} \\ &= (D \cdot B_3) = (\alpha_3 + \beta_3) d_1 + \alpha_3 d_2 \geq 2d_1 + d_2 \geq 2d_1. \end{aligned}$$

Therefore, all the equalities here must hold, which gives

$$d_1 = m_1 = m_2, \quad \alpha_3 = \beta_3 = 1, \quad \text{and} \quad d_2 = 0.$$

But then $D \in |d_1 C_1|$, and $d_1 = d_1 + d_2 \geq 2$. By Lemma 2.24, we must have $d_1 > m_1$ and $d_1 > m_2$, contradiction. This completes the proof of the claim. $\square$

From the claim, we see that $D \cap B_3 = \{p_{i,3}\}$ for some $i \in \{1, 2\}$. Notice that D must be tangent to B_3 at $p_{i,3}$ because

$$(D \cdot B_3) = (\alpha_3 + \beta_3) d_1 + \alpha_3 d_2 \geq 2d_1 + d_2 > d_1 \geq \mathrm{mult}_{p_{i,3}}(D).$$

Thus, D must be transverse to B_i at $p_{i,3}$. Then, D must meet B_i at a further point. Indeed, we have

$$(D \cdot B_i) = (\alpha_i + \beta_i)d_1 + \alpha_i d_2 \geq d_1 + d_2 \geq d_1 \geq \operatorname{mult}_{p_{i,3}}(D) = (D \cdot B_i)_{p_{i,3}}.$$

If $D \cap B_i = \{p_{i,3}\}$, then all the equalities here must hold, which gives $\alpha_i = 1$, $\beta_i = 0$, $d_2 = 0$ and $\operatorname{mult}_{p_{i,3}}(D) = d_1 = d_1 + d_2 \geq 2$. But then $D \in |d_1 C_1|$, and by Lemma 2.24 we know $\operatorname{mult}_{p_{i,3}}(D) < d_1$, contradiction.

Since D must meet the other component B_j (where $j \neq i, 3$), we know there exists a point $p_{1,2} \in B_1 \cap B_2$ such that $D \cap B_j = \{p_{1,2}\}$ and $D \cap B = \{p_{1,2}, p_{i,3}\}$.

Now, B_j and D meet only at $p_{1,2}$. We know that D cannot be transverse to B_j at $p_{1,2}$ for the same reason as before. Thus, D must be transverse to B_i at $p_{1,2}$.

Hence, $D \cap B_i = \{p_{1,2}, p_{i,3}\}$, and they intersect transversally at both of these points. By Lemma 2.23, we know $d_1 \geq \operatorname{mult}_{p_{1,2}}(D)$ and $d_1 \geq \operatorname{mult}_{p_{i,3}}(D)$. We obtain

$$\begin{aligned} 2d_1 &\geq \operatorname{mult}_{p_{1,2}}(D) + \operatorname{mult}_{p_{i,3}}(D) \\ &= (D \cdot B_i)_{p_{1,2}} + (D \cdot B_i)_{p_{i,3}} \\ &= (D \cdot B_i) \\ &= (\alpha_i + \beta_i)d_1 + \alpha_i d_2 \\ &\geq d_1 + d_2. \end{aligned}$$

By assumption, we have $\alpha_i \geq 1$ and $\beta_i \geq 0$. Thus, we must have $\alpha_i = 1$, $\beta_i = 0$ and $d_2 \leq d_1$. Therefore, B_i is contained in $|C_1|$. By the assumption $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2$, we must have $B_1 \in |C_1|$. This proves part (1) of the proposition.

Without loss of generality, we may assume $i = 1$. Then, $D \cap B_1 = \{p_{1,2}, p_{1,3}\}$. Now, we have

$$\operatorname{mult}_{p_{1,2}}(D) + \operatorname{mult}_{p_{1,3}}(D) = (D \cdot B_1)_{p_{1,2}} + (D \cdot B_1)_{p_{1,3}} = (D \cdot B_1) = d_1 + d_2.$$

Let $m \coloneqq \operatorname{mult}_{p_{1,2}}(D)$. Then, $m \leq d_1$ by Lemma 2.23, and we have

$$D \in \operatorname{Hyp}^{m}_{d_1 C_1 + d_2 f}(B_2, p_{1,2}) \cap \operatorname{Hyp}^{d_1 + d_2 - m}_{d_1 C_1 + d_2 f}(B_3, p_{1,3}).$$

Case 1, $d_2 = 0$:

In this case, we have $d_1 \geq 2$, $m < d_1$ by Lemma 2.24, and

$$D \in \operatorname{Hyp}^{m}_{d_1 C_1}(B_2, p_{1,2}) \cap \operatorname{Hyp}^{d_1 - m}_{d_1 C_1}(B_3, p_{1,3}).$$

By Theorem 2.11, we have

$$\delta_D(p_{1,2}) \geq \frac{(m-1)\left((D \cdot B_2) - 1\right)}{2} = \frac{(m-1)\left(\alpha_2 d_1 + \beta_2 d_1 - 1\right)}{2}$$

and

$$\delta_D(p_{1,3}) \geq \frac{(d_1 - m - 1)\left((D \cdot B_3) - 1\right)}{2} = \frac{(d_1 - m - 1)\left(\alpha_3 d_1 + \beta_3 d_1 - 1\right)}{2}.$$

From these, we obtain

$$
\begin{aligned}
0 \le p_a(D^\nu) &\le p_a(D) - \delta_D(p_{1,2}) - \delta_D(p_{1,3}) \\
&\le \frac{1}{2}(d_1-1)(d_1-2) \\
&\quad - \frac{1}{2}(m-1)(\alpha_2 d_1 + \beta_2 d_1 - 1) \\
&\quad - \frac{1}{2}(d_1-m-1)(\alpha_3 d_1 + \beta_3 d_1 - 1).
\end{aligned} \tag{1}
$$

As $\alpha_2 \ge 1$ and $\beta_2 \ge 0$, we have

$$
\begin{aligned}
0 \le p_a(D^\nu) &\le \frac{1}{2}(d_1-1)(d_1-2) - \frac{1}{2}(m-1)(d_1-1) \\
&\quad - \frac{1}{2}(d_1-m-1)(\alpha_3 d_1 + \beta_3 d_1 - 1) \\
&= \frac{1}{2}(d_1-1)(d_1-m-1) \\
&\quad - \frac{1}{2}(d_1-m-1)(\alpha_3 d_1 + \beta_3 d_1 - 1) \\
&= -\frac{1}{2}(d_1-m-1)(\alpha_3+\beta_3-1)d_1 \\
&\le 0.
\end{aligned}
$$

Therefore, the equalities here must hold, which implies that $d_1 = m+1$ as claimed in part (2) of the proposition. Substituting this back into the inequality (1), we obtain

$$
\begin{aligned}
0 \le p_a(D^\nu) &\le \frac{1}{2}m(m-1) - \frac{1}{2}(m-1)\left((\alpha_2+\beta_2)m + (\alpha_2+\beta_2-1)\right) \\
&= -\frac{1}{2}(m-1)^2(\alpha_2+\beta_2-1) \le 0.
\end{aligned}
$$

Therefore, all the equalities must hold and $p_a(D^\nu) = 0$.

If $m \ge 2$ (equivalently, $d_1 \ge 3$), we must have $\alpha_2 = 1$ and $\beta_2 = 0$, that is, when $d_1 \ge 3$, $\mathrm{Hyp}_{d_1 C_1}(B,2) \ne \emptyset$ only if $\{B_1, B_2\} \subset |C_1|$. Moreover, if $\{B_1, B_2\} \subset |C_1|$, we also need to consider those curves in $\mathrm{Hyp}_{d_1 C_1}(B,2)$ that intersect B_2 at two points. To do this, we may switch B_1 and B_2, then by symmetry, the preceding arguments remain valid.

So, to prove the last statement of part (2) of the proposition, we first need to study the cardinality of $\mathrm{Hyp}_{2C_1}(B,2)$.

Claim 4.5. With $p_{1,2}$ and $p_{1,3}$ fixed, there is at most one integral curve in $\mathrm{Hyp}_{2C_1}(B,2)$ that contains both $p_{1,2}$ and $p_{1,3}$

Proof of the Claim. Let D and D' be two such curves. Then as we have seen before, we have $D \cap B_2 = \{p_{1,2}\} = D' \cap B_2$ and $D \cap B_3 = \{p_{1,3}\} = D' \cap B_3$.

We obtain

$$(D\cdot B_2)_{p_{1,2}} = (D\cdot B_2) = 2(\alpha_2+\beta_2) = (D'\cdot B_2) = (D'\cdot B_2)_{p_{1,2}}$$

and

$$(D\cdot B_3)_{p_{1,3}} = (D\cdot B_3) = d_1(\alpha_3+\beta_3) = (D'\cdot B_3) = (D'\cdot B_3)_{p_{1,3}}.$$

Since $(D'\cdot D)=4$, we must have

$$(D'\cdot D)_{p_{1,3}} < 4 \leq 2(\alpha_3+\beta_3) = (D'\cdot B_3)_{p_{1,3}} = (D\cdot B_3)_{p_{1,3}}.$$

Note that $p_{1,3}$ is a smooth point on all three curves D, D' and B_3. But then, by Theorem 2.13, the smallest two among

$$(D'\cdot D)_{p_{1,3}},\quad (D'\cdot B_3)_{p_{1,3}},\quad (D\cdot B_3)_{p_{1,3}}$$

are equal, contradiction.

□

If $\{B_1,B_2\}\cap|C_1|=\{B_1\}$, then since there are only $(B_1\cdot B_2)=\alpha_2+\beta_2$ choices for $p_{1,2}$ and $(B_1\cdot B_3)=\alpha_3+\beta_3$ choices for $p_{1,3}$, we obtain

$$|\mathrm{Hyp}_{2C_1}(B,2)| \leq (\alpha_2+\beta_2)(\alpha_3+\beta_3).$$

If $\{B_1,B_2\}\subset|C_1|$, then $B_1\cap B_2=\{p_{1,2}\}$. As mentioned before, we also need to consider those curves in $\mathrm{Hyp}_{2C_1}(B,2)$ that intersect B_2 at two points. By symmetry we obtain

$$|\mathrm{Hyp}_{2C_1}(B,2)| \leq (B_1\cdot B_3)+(B_2\cdot B_3) = 2(\alpha_3+\beta_3).$$

In both cases, $|\mathrm{Hyp}_{2C_1}(B,2)|$ is effectively bounded.

Finally, if $\{B_1,B_2\}\cap|C_1|=\{B_1\}$, we have seen that $|\mathrm{Hyp}_{d_1C_1}(B,2)|=\emptyset$ for all $d_1>2$. If $\{B_1,B_2\}\subset|C_1|$, then by contracting C_0 we obtain the blow-up morphism $\pi\colon \mathbb{F}_1\to\mathbb{P}^2$. Note that $C_0\cap B_1=\emptyset=C_0\cap B_2$. Therefore, the plane curve $\pi(B)=\pi(B_1)\cup\pi(B_2)\cup\pi(B_3)$ is still a $3C$-curve, $\pi(D)\in\mathrm{Hyp}(\mathbb{P}^2,\pi(B),2)$, and $\pi(D)\cap\pi(B_j)=\{\pi(p_{1,j})\}$ for $j=2,3$. We also have

$$\pi(B_1),\pi(B_2)\in|\mathcal{O}(1)|,\quad \pi(B_3)\in|\mathcal{O}(\alpha_3+\beta_3)|.$$

Hence $\deg\pi(B)\geq 4$; we reduce the problem to the cases on $\mathbb{P}^2$, which are already studied in [CT25a, Proposition 3.2.1, Theorem 3.3.1, Theorem 3.3.2]. From these, we conclude the proof of part (2) of the proposition.

Case 2, $d_2>0$:

Recall that we assumed that B_1 intersects D transversally at $p_{1,2}$ and $p_{1,3}$, and we have

$$\mathrm{mult}_{p_{1,2}}(D)+\mathrm{mult}_{p_{1,3}}(D) = (D\cdot B_1)_{p_{1,2}}+(D\cdot B_1)_{p_{1,3}} = (D\cdot B_1) = (D\cdot C_1) = d_1+d_2,$$

$m\coloneqq\mathrm{mult}_{p_{1,2}}(D)\leq d_1$, and

$$D\in\mathrm{Hyp}_{d_1C_1+d_2f}(B,2)\subseteq\mathrm{Hyp}^m_{d_1C_1+d_2f}(B_2,p_{1,2})\cap\mathrm{Hyp}^{d_1+d_2-m}_{d_1C_1+d_2f}(B_3,p_{1,3}).$$

By Theorem 2.11, we have

$$\delta_D(p_{1,2}) \geq \frac{(m-1)\left((D\cdot B_2)-1\right)}{2} = \frac{(m-1)\left(\alpha_2 d_1+\alpha_2 d_2+\beta_2 d_1-1\right)}{2}$$

and

$$\begin{aligned}\delta_D(p_{1,3}) &\geq \frac{(d_1+d_2-m-1)\left((D\cdot B_3)-1\right)}{2}\\ &= \frac{(d_1+d_2-m-1)\left(\alpha_3 d_1+\alpha_3 d_2+\beta_3 d_1-1\right)}{2}.\end{aligned}$$

From these, we obtain

$$\begin{aligned}0 \leq p_a(D^\nu) &\leq p_a(D)-\delta_D(p_{1,2})-\delta_D(p_{2,3})\\ &\leq \frac{1}{2}(d_1-1)(d_1+2d_2-2)\\ &\quad -\frac{1}{2}(m-1)\left(\alpha_2 d_1+\alpha_2 d_2+\beta_2 d_1-1\right)\\ &\quad -\frac{1}{2}(d_1+d_2-m-1)\left(\alpha_3 d_1+\alpha_3 d_2+\beta_3 d_1-1\right).\end{aligned} \tag{2}$$

As $\alpha_2 \geq 1, \beta_2 \geq 0$, we get

$$\begin{aligned}0 \leq p_a(D^\nu) &\leq \frac{1}{2}(d_1-1)(d_1+2d_2-2)\\ &\quad -\frac{1}{2}(m-1)(d_1+d_2-1)\\ &\quad -\frac{1}{2}(d_1+d_2-m-1)((\alpha_3+\beta_3)d_1+\alpha_3 d_2-1)\\ &= \frac{1}{2}\left(-(\alpha_3+\beta_3-1)d_1^2-(2\alpha_3+\beta_3-2)d_1d_2-\alpha_3 d_2^2\right)\\ &\quad +\frac{1}{2}\left((\alpha_3+\beta_3-1)(m+1)d_1+((\alpha_3-1)m+\alpha_3)d_2\right)\\ &= -\frac{1}{2}\left[(\alpha_3+\beta_3-1)d_1+(\alpha_3-1)d_2\right](d_1-m)\\ &\quad -\frac{1}{2}\left[(\alpha_3+\beta_3-1)d_1+\alpha_3 d_2\right](d_2-1).\end{aligned} \tag{3}$$

Note that $d_1 \geq m$ by Lemma 2.23 and $d_2 \geq 1$ by assumption, and recalling that $\alpha_3 \geq 1$ and $\beta_3 \geq 1$. For the coefficient of (d_1-m) in the last part of inequality (3) above, we get

$$(\alpha_3+\beta_3-1)d_1+(\alpha_3-1)d_2 \geq d_1 > 0;$$

and for the coefficient of (d_2-1) in the last part of inequality (3) above, we get

$$(\alpha_3+\beta_3-1)d_1+\alpha_3 d_2 \geq d_1+d_2 > 0.$$

Therefore, all the equalities above in the inequality (3) must hold. We obtain

$$d_1 = m, \quad d_2 = 1, \quad \text{and} \quad p_a(D^\nu) = 0.$$

Substituting these back into the inequality (2), we obtain

$$\begin{aligned} 0 = p_a(D^\nu) &\leq \frac{1}{2}(m-1)m - \frac{1}{2}(m-1)((\alpha_2+\beta_2)m + \alpha_2 - 1) \\ &= -\frac{1}{2}(m-1)((\alpha_2+\beta_2-1)m + \alpha_2 - 1) \\ &\leq 0. \end{aligned}$$

Therefore, all the equalities here also must hold. If $m > 1$, we must have $\alpha_2 = 1$ and $\beta_2 = 0$. If $\{B_1, B_2\} \subset |C_1|$, we also need to consider those curves in $\mathrm{Hyp}_{d_1C_1+f}(B, 2)$ that intersect B_2 at two points. To do this, we may switch B_1 and B_2, then by symmetry, the preceding arguments remain valid. Hence we conclude the proof of part (3) of the proposition.

Since integral curves in $|C_0|$, $|C_1|$ and $|f|$ are already rational, the last statement $\mathrm{Hyp}(B, 2) = \mathcal{E}(B)$ immediately follows from part (2) and part (3). $\square$

Now, we will show that $\mathcal{E}(B)$ in Proposition 4.3 is a finite set.

Theorem 4.6. *Consider $\mathbb{F}_1$. Let B be as in Proposition 4.1. Then:*

(a) If $\{B_1, B_2\} \not\subset |C_1|$, then $\mathcal{E}(B)$ is finite and is effectively bounded in terms of α_i's and β_i's.

(b) If $\{B_1, B_2\} \subset |C_1|$, then $|\mathcal{E}(B)| < \infty$.

Proof. Combining Proposition 4.1 and part (2) of Proposition 4.3, we see that $\left|\bigcup_{d_1 \geq 1} \mathrm{Hyp}_{d_1C_1}(B, 2)\right|$ is finite and effectively bounded in terms of α_i's and β_i's.

By Proposition 4.3, we see it remains to study the cardinality of $\mathrm{Hyp}_{C_0}(B, 2)$, $\mathrm{Hyp}_f(B, 2)$, and $\mathrm{Hyp}_{d_1C_1+f}(B, 2)$ with $d_1 \geq 1$.

Since $|C_0| = \{C_0\}$, we have $|\mathrm{Hyp}_{C_0}(B, 2)| \leq 1$. By the same argument as in the proof of Theorem 3.3 for the $\mathbb{F}_0$ case, we see that any integral curve $D \in \mathrm{Hyp}_f(B, 2)$ contains some point in N. Therefore, we obtain

$$|\mathrm{Hyp}_f(B, 2)| \leq |N| = \sum_{i<j} (B_i \cdot B_j).$$

Now, it remains to study $\mathrm{Hyp}_{d_1C_1+f}(B, 2)$, where $d_1 \geq 1$. By part (3) of Proposition 4.3, we know that if $\mathrm{Hyp}_{d_1C_1+f}(B, 2) \neq \emptyset$ for some $d_1 \geq 1$, then $B_1 \in |C_1|$.

For $D \in \mathrm{Hyp}_{d_1C_1+f}(B, 2)$, we first consider the case that D intersects B_1 at two points. Then, as in the proof of part (3) of Proposition 4.3, we have $D \cap B = \{p_{1,2}, p_{1,3}\}$ for some points $p_{1,2} \in B_1 \cap B_2$ and $p_{1,3} \in B_1 \cap B_3$, $D \cap B_2 = \{p_{1,2}\}$, $D \cap B_3 = \{p_{1,3}\}$, and D intersects B_1 transversally at both $p_{1,2}$ and $p_{1,3}$. Moreover, we know that $p_{1,2}$ is a unibranch d_1-fold point of D, and B_2 is tangent to D at $p_{1,2}$ since $(B_2 \cdot D)_{p_{1,2}} = (B_2 \cdot D) \geq d_1 + 1 > d_1$.

Claim 4.7. Fix $p_{1,2}$ and $p_{1,3}$. For any $d_1 \geq 1$, there is at most one integral curve in $\mathrm{Hyp}_{d_1C_1+f}(B,2)$ that contains both $p_{1,2}$ and $p_{1,3}$. Moreover, if $D_1 \in \mathrm{Hyp}_{m_1C_1+f}(B,2)$ and $D_2 \in \mathrm{Hyp}_{m_2C_1+f}(B,2)$ are two integral curves containing $p_{1,2}$ and $p_{1,3}$ with $m_1 < m_2$, then

$$(\alpha_3 + \beta_3)m_1 + \alpha_3 \leq m_2.$$

Proof of the Claim. Suppose there are two curves $D, D' \in \mathrm{Hyp}_{d_1C_1+f}(B,2)$ containing $p_{1,2}$ and $p_{1,3}$. Then, both of them intersect B_2 only at $p_{1,2}$ and intersect B_3 only at $p_{1,3}$, $\mathrm{mult}_{p_{1,2}}(D) = \mathrm{mult}_{p_{1,2}}(D') = d_1$, and $p_{1,3}$ is a smooth point on both of them. We obtain

$$(D \cdot B_3)_{p_{1,3}} = (D \cdot B_3) = d_1(\alpha_3 + \beta_3) + \alpha_3 = (D' \cdot B_3) = (D' \cdot B_3)_{p_{1,3}},$$

and

$$(D \cdot B_2)_{p_{1,2}} = (D \cdot B_2) = d_1 + 1 = (D' \cdot B_2) = (D' \cdot B_2)_{p_{1,2}}.$$

By Theorem 2.13, we obtain

$$\frac{(D \cdot D')_{p_{1,2}}}{d_1^2} \geq \frac{(D \cdot B_2)_{p_{1,2}}}{d_1} = \frac{(D' \cdot B_2)_{p_{1,2}}}{d_1} = \frac{d_1 + 1}{d_1}.$$

Therefore,

$$\begin{aligned}
(D \cdot D')_{p_{1,3}} &\leq (D \cdot D') - (D \cdot D')_{p_{1,2}} \\
&\leq (d_1^2 + 2d_1) - (d_1^2 + d_1) \\
&= d_1 \\
&< d_1(\alpha_3 + \beta_3) + \alpha_3 \\
&= (D \cdot B_3)_{p_{1,3}} = (D' \cdot B_3)_{p_{1,3}}.
\end{aligned}$$

But then, by Theorem 2.13, the smallest two among

$$(D \cdot D')_{p_{1,3}}, \quad (D \cdot B_3)_{p_{1,3}}, \quad (D' \cdot B_3)_{p_{1,3}}$$

are equal, contradiction.

Now, let $D_1 \in \mathrm{Hyp}_{m_1C_1+f}(B,2)$ and $D_2 \in \mathrm{Hyp}_{m_2C_1+f}(B,2)$ be two curves containing $p_{1,2}$ and $p_{1,3}$ with $m_1 < m_2$. Then both D_1 and D_2 are hypertangent to B_2 at $p_{1,2}$. By Theorem 2.13, we know that the smallest two among

$$\frac{(D_1 \cdot D_2)_{p_{1,2}}}{m_1m_2}, \quad \frac{(D_1 \cdot B_2)_{p_{1,2}}}{\mathrm{mult}_{p_{1,2}}(D_1)} = \frac{m_1 + 1}{m_1}, \quad \frac{(D_2 \cdot B_2)_{p_{1,2}}}{\mathrm{mult}_{p_{1,2}}(D_2)} = \frac{m_2 + 1}{m_2}$$

are equal. Hence,

$$\frac{(D_1 \cdot D_2)_{p_{1,2}}}{m_1m_2} = \frac{m_2 + 1}{m_2}.$$

Moreover, both D_1 and D_2 are hypertangent to B_3 at $p_{1,3}$. Since

$$(D_1 \cdot D_2) = ((m_1C_1 + f) \cdot (m_2C_1 + f)) = m_1m_2 + m_1 + m_2,$$

we have

$$(D_1 \cdot D_2)_{p_{1,3}} \leq (D_1 \cdot D_2) - (D_1 \cdot D_2)_{p_{1,2}} = (m_1m_2 + m_1 + m_2) - (m_1m_2 + m_1) = m_2.$$

Furthermore, we have

$$(B_3 \cdot D_1)_{p_{1,3}} = (B_3 \cdot D_1) = (\alpha_3 + \beta_3)m_1 + \alpha_3$$

and

$$(B_3 \cdot D_2)_{p_{1,3}} = (B_3 \cdot D_2) = (\alpha_3 + \beta_3)m_2 + \alpha_3.$$

Then

$$(B_3 \cdot D_1)_{p_{1,3}} = (\alpha_3 + \beta_3)m_1 + \alpha_3 < (\alpha_3 + \beta_3)m_2 + \alpha_3 = (B_3 \cdot D_2)_{p_{1,3}}.$$

Note that $p_{1,3}$ is a smooth point on all three curves D, D' and B_3. By Theorem 2.13, we must have

$$(\alpha_3 + \beta_3)m_1 + \alpha_3 = (B_3 \cdot D_1)_{p_{1,3}} = (D_1 \cdot D_2)_{p_{1,3}} \leq m_2.$$

This completes the proof of the claim. $\square$

If $\{B_1, B_2\} \cap |C_1| = \{B_1\}$, then, by part (3) of Proposition 4.3, we obtain $\mathrm{Hyp}_{d_1C_1+f} = \emptyset$ for any $d_1 \geq 2$. Moreover, in this case, note that there are at most $(B_1 \cdot B_2)$ choices for $p_{1,2}$ and at most $(B_1 \cdot B_3)$ choices for $p_{1,3}$, and by Claim 4.7, we know

$$|\mathrm{Hyp}_{C_1+f}(B, 2)| \leq (B_1 \cdot B_2)(B_1 \cdot B_3) = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3).$$

Combining Proposition 4.1 and Proposition 4.3, we obtain part (a) of the theorem.

If $\{B_1, B_2\} \subset |C_1|$, consider the blow-up morphism $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$ obtained by contracting C_0. We have

$$\deg \pi(B_1) = \deg \pi(B_2) = 1, \quad \deg \pi(B_3) = \alpha_3 + \beta_3 \geq 2, \quad \deg \pi(D) = d_1 + 1,$$

and $\pi(D) \cap \pi(B) = \{q, \pi(p_{1,2}), \pi(p_{1,3})\}$, where $q = \pi(C_0)$ is the blow-up point.

Then, $\pi(B)$ is still a $3C$-curve. Indeed, in $\mathbb{F}_1$, $B_1, B_2 \in |C_1|$ ensures that B_1 and B_2 do not intersect C_0. Now, the Corvaja–Zannier theorem [**CZ13**, Theorem 1] tells us

$$d_1 + 1 = \deg \pi(D) \leq \gamma(B),$$

for some constant positive integer $\gamma(B)$ determined by B.

Let $I_1 = \{m_1, m_2, \ldots, m_n, \ldots\} \subset \mathbb{N}$ be the set of all positive integers such that for each $m_i \in I_1$, there exists a curve in $\mathrm{Hyp}_{m_iC_1+f}(B, 2)$ containing both $p_{1,2}$ and $p_{1,3}$. Then, by Claim 4.7, for all $i \geq 2$, we obtain

$$m_i \geq (\alpha_3 + \beta_3)m_{i-1} + \alpha_3 > (\alpha_3 + \beta_3)m_{i-1}.$$

By induction, we get

$$m_i > (\alpha_3 + \beta_3)^{i-1}m_1, \quad \forall i \geq 2.$$

Hence,

$$(\alpha_3 + \beta_3)^{i-1}m_1 < \gamma(B) - 1, \quad \forall i \geq 2.$$

If $|I_1| \geq 2$, this implies

$$|I_1| = \max_{i\in\mathbb{N},\ m_i\in I}\{i\} < 1 + \log_{(\alpha_3+\beta_3)}\left(\frac{\gamma(B)-1}{m_1}\right) \leq 1 + \log_{(\alpha_3+\beta_3)}(\gamma(B)-1).$$

Therefore, we conclude that I_1 is a finite set, and

$$|I_1| \leq 1 + \lfloor \log_{(\alpha_3+\beta_3)}(\gamma(B)-1) \rfloor.$$

In the case $\{B_1, B_2\} \subset |C_1|$, we also need to consider those curves in $\mathrm{Hyp}_{d_1C_1+f}(B,2)$ which intersect B_2 at two points. If $D \in \mathrm{Hyp}_{d_1C_1+f}(B,2)$ is such a curve, then $D \cap B_2 = \{p_{1,2}, p_{2,3}\}$, where $\{p_{1,2}\} = B_1 \cap B_2$ and $p_{2,3} \in B_2 \cap B_3$. By symmetry, after switching B_1 and B_2, the preceding arguments remain valid. Let $I_2 = \{m'_1, m'_2, \dots, m'_n, \dots\} \subset \mathbb{N}$ be the set of all positive integers such that for each $m'_i \in I_2$, there exists a curve in $\mathrm{Hyp}_{m'_iC_1+f}(B,2)$ containing both $p_{1,2}$ and $p_{2,3}$, then, by symmetry, we also have

$$|I_2| \leq 1 + \lfloor \log_{(\alpha_3+\beta_3)}(\gamma(B)-1) \rfloor.$$

Now, since $\{p_{1,2}\} = B_1 \cap B_2$ and there are only $\alpha_3 + \beta_3$ choices for $p_{1,3}$ and $\alpha_3 + \beta_3$ choices for $p_{2,3}$, combining Proposition 4.3, we conclude that $\mathcal{E}(B)$ is a finite set, which gives part (b) of the theorem and completes the proof. □

EXAMPLE 4.8. In the case where $B_1, B_2 \in |C_1|$, if $B_3 \in |C_1 + f|$ and B is in general position, with all the notations being the same as above in the proof of Theorem 4.6, then, by another result of Corvaja–Zannier [CZ08, Theorem 1.1], we have $\gamma(B) \leq 2^{15} \cdot 35$. Then, we obtain

$$|I_j| < 1+\log_2\left(2^{15}\cdot 35 - 1\right) < 1+\log_2\left(2^{15}\cdot 35\right) = 1+15+\log_2(35) < 22, \quad j = 1, 2.$$

PROPOSITION 4.9. *Consider $\mathbb{F}_1$. Let B be as in Proposition 4.1.*

If $\alpha_i \geq 3$ for all $i \in \{1,2,3\}$, $\beta_l \geq 3$ for some $l \in \{1,2,3\}$, and B is general, then

$$\mathcal{E}(B) = \emptyset.$$

PROOF. The statement follows by combining Proposition 4.3 and Theorem 4.6 with the argument from the proof of Proposition 3.8. □

Finally, we provide an example that if the assumption on the $3C$-curve B—that none of its three components lies in $|f|$ or is C_0—is not satisfied, then $\mathrm{Hyp}(B,2) \neq \mathcal{E}(B)$:

EXAMPLE 4.10. Let $D \subset \mathbb{P}^2$ be a smooth cubic curve with a flex fixed as the neutral element for the group structure on it, and let $p \in D$ be a 9-torsion point which is not a flex. By [MPS25, Theorem 1.3, Proposition 2.2], there exists a unique pencil $\mathcal{V}$ containing D such that p is the only base point of the pencil.

Let q be a flex of D, and let L be the tangent line to D at q.

CLAIM 4.11. There exists a Zariski-open dense subset $U \subset \mathcal{V}$ where every member intersects L transversally.

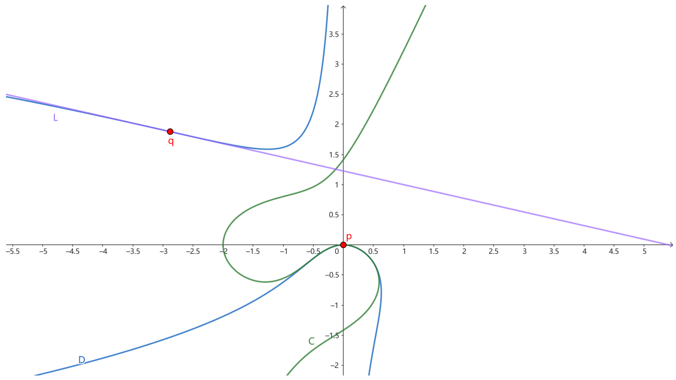

FIGURE 1. Example 4.10

Let C be a smooth cubic curve in U; then $C \cap D = \{p\}$ by construction. Blow up a point x in $L \setminus (C \cup D)$. In the blow-up surface $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$, denote the exceptional line by E. The curve $B \coloneqq \tilde{L} \cup E \cup \tilde{C}$ is a $3C$-curve, and by Proposition 2.26 we know

$$K_{\mathbb{F}_1} + B \sim -2C_1 - f + f + C_0 + 3C_1 \sim C_1 + C_0 \sim 2C_1 - f$$

is a big divisor on $\mathbb{F}_1$. However, $\tilde{D} \in \mathrm{Hyp}(B, 2)$ has geometric genus 1. Therefore, $\mathrm{Hyp}(B, 2) \neq \mathcal{E}(B)$ in this case.

It remains to prove the claim. By construction, the pencil $\mathcal{V}$ has only one base point p which is not contained in the line L. Therefore, the restriction of $\mathcal{V}$ to L—denoted as $\mathcal{V}_L$—is a base-point-free linear system on L. By Bertini's theorem ([**Har77**, Chapter III, Corollary 10.9, page 274]), there exists a Zariski-open dense subset $U' \subseteq \mathcal{V}_L$ where every member is smooth as a closed subscheme of L. This implies that there exists a Zariski-open dense subset $U \subseteq \mathcal{V}$ where every member intersects L transversally. Thus, the claim is established.

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Department of Mathematics and Physics, Roma Tre University, Largo San Leonardo Murialdo, I-00146, Rome, Italy
Email address: wei.chen@uniroma3.it, weichen97.ag@gmail.com