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AUTOMORPHISMS OF PRIME POWER ORDER OF WEIGHTED HYPERSURFACES

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ABSTRACT. We study automorphisms of quasi-smooth hypersurfaces in weighted projective spaces, extending classical results for smooth hypersurfaces in projective space to the weighted setting. We establish effective criteria for when a power of a prime number can occur as the order of an automorphism, and we derive explicit bounds on the possible prime orders. A key role is played by a weighted analogue of the classical Klein hypersurface, which we show realizes the maximal prime order of an automorphism under suitable arithmetic conditions. Our results generalize earlier work by González-Aguilera and Liendo.

INTRODUCTION

Smooth hypersurfaces in complex projective space are fundamental objects in algebraic geometry, and understanding their automorphism groups is a classical problem. For a smooth hypersurface $X \subset \mathbf{P}^{n+1}$ of dimension n and degree d, it is well known that if $d \geq 3$ and $(n,d) \neq (1,3),(2,4)$, then the automorphism group $\mathrm{Aut}(X)$ is finite, and every automorphism of X is induced by a projective linear transformation of the ambient projective space [MM64]. Typically, a general hypersurface has a trivial automorphism group, but certain special hypersurfaces exhibit nontrivial automorphism groups. This naturally leads to the following question: *which finite groups can occur as subgroups of* $\mathrm{Aut}(X)$*?*

A first step towards addressing this question was taken in [GAL11], where all cyclic groups of prime order acting faithfully on smooth cubic hypersurfaces $X \subset \mathbf{P}^{n+1}$ were classified. This result was soon generalized in [GAL13] to classify cyclic groups of prime power order acting faithfully on smooth hypersurfaces of any degree $d \geq 3$. More recently, a complete classification of all abelian groups acting faithfully on smooth hypersurfaces of degree $d \geq 3$ was obtained in [Zhe22]; see also [GALM22]. Full classifications are also available for specific cases, such as smooth cubic threefolds [WY19], smooth quintic threefolds [OY19], smooth cubic fivefolds and fourfolds [YYZ24], and symplectic automorphisms of smooth cubic fourfolds [LZ22].

Finally, it is worth noting that two recent and independent works [EL25, YYZ25] showed that, with finitely many exceptions, the smooth hypersurface of degree $d \geq 3$ in $\mathbf{P}^{n+1}$ that admits the largest automorphism group is the Fermat hypersurface, i.e., the zero locus of the homogeneous polynomial

$$F = x_0^d + x_1^d + \cdots + x_n^d + x_{n+1}^d = 0\,.$$

In this paper, we extend the methods and results of [GAL11, GAL13, GALM22, GALMVL24] to the setting of quasi-smooth hypersurfaces in weighted projective spaces. Let $\mathbf{P}_a^{n+1}$ denote the weighted projective space associated with the weights $a = (a_0, a_1, \ldots, a_{n+1})$, where each $a_i \in \mathbf{Z}_{>0}$. A hypersurface $X = V(F) \subset \mathbf{P}_a^{n+1}$ defined by a homogeneous polynomial F of degree d is called quasi-smooth if its affine cone $\{x \in \mathbf{A}^{n+2} \mid F(x) = 0\}$ is smooth away from the origin [Dan91, Dol81, BC94].

The automorphism group $\mathrm{Aut}(X)$ of a quasi-smooth hypersurface in weighted projective space is not necessarily induced by automorphisms of the ambient space. However, under mild conditions analogous to those in the classical setting [MM64], it coincides with the group of automorphisms induced by $\mathrm{Aut}(\mathbf{P}_a^{n+1})$ [Ess24, Theorem 2.1]. The group $\mathrm{Aut}(\mathbf{P}_a^{n+1})$ itself admits a natural description in terms of the group of equivariant automorphisms of the

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affine space $\mathbf{A}^{n+2}$ modulo the image of the diagonal $\mathbf{G}_\mathrm{m}$-action defined by the weights a, where $\mathbf{G}_\mathrm{m}$ stands for the multiplicative group of the base field $\mathbf{C}$ [AA89].

Let $\mathbf{P}_a^{n+1}$ be the weighted projective space with weights $a = (a_0, a_1, \ldots, a_{n+1})$, where each $a_i \in \mathbf{Z}_{>0}$, and let $d \geq 3$. Let also p, r be positive integers with p prime. The main technical results of this paper include a necessary condition for a cyclic group of order p^r to act faithfully on a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$, see Proposition 2.2; as well as a sufficient condition for such an action under additional hypotheses, see Proposition 2.5.

In the special case where each weight a_i divides d, we establish in Theorem 2.6 a complete criterion for the cyclic group of order p prime that can act faithfully on a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$. This generalizes the result of [GAL13]. We apply this result to derive an explicit bound on the cyclic groups of prime order p that can act faithfully on a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$, see Corollary 2.8. The result in Theorem 2.6 not only extends the classical case of projective space [GAL13, Proposition 2.2], but also applies to various geometric settings in which each a_i divides d, including the study of K-stability and automorphism groups of weighted Fano hypersurfaces [ST24], and the classification of Fano threefolds containing a smooth rational surface with ample normal bundle [CF93].

We then turn our attention to quasi-smooth hypersurfaces in weighted projective spaces in the complementary case where each weight a_i is relatively prime to d. Under this assumption, we establish the following bound: if a cyclic group of prime order p acts faithfully on a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$, then

$$p < \left(\frac{\max(a)}{d - \max(a)}\right) \prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t}\right),$$

where $\max(a)$ denotes the maximum of the weights, see Corollary 3.1. This result generalizes [GAL13, Corollary 2.4]. Let now

$$p = \frac{1}{d}\left[\prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t}\right) + (-1)^{n+1}\right]$$

and assume that p is a prime number. In Theorem 3.5, we prove that the cyclic group of order p is the largest cyclic group of prime order that can act faithfully on a quasi-smooth hypersurface X of degree d in $\mathbf{P}_a^{n+1}$, and that X is isomorphic to a weighted Klein hypersurface, that is, a weighted hypersurface defined as the zero locus of a homogeneous polynomial of the form

$$K = x_0^{m_0} x_1 + x_1^{m_1} x_2 + \ldots + x_n^{m_n} x_{n+1} + x_{n+1}^{m_{n+1}} x_0 \,.$$

The paper is organized as follows. In Section 1 we review basic definitions concerning weighted projective spaces, quasi-smoothness, and automorphism groups of weighted quasi-smooth hypersurfaces. In Section 2, we establish our main result: a criterion for the existence of automorphisms of prime power order in the quasi-smooth weighted setting. Finally, in Section 3 we investigate the structure of automorphisms of maximal possible prime order and their relation to weighted Klein hypersurfaces.

1. Automorphism groups of weighted projective spaces and quasi-smooth hypersurfaces

Letting $n \geq 1$, we fix $a = (a_0, a_1, \ldots, a_{n+1}) \in \mathbf{Z}_{>0}^{n+2}$ such that $\gcd(a_0, a_1, \ldots, a_{n+1}) = 1$. Let $\mathbf{A}^{n+2} = \operatorname{Spec} \mathbf{C}[x_0, \ldots, x_{n+1}]$, and consider the $\mathbf{G}_\mathrm{m}$-action

$$\alpha \colon \mathbf{G}_\mathrm{m} \times \mathbf{A}^{n+2} \to \mathbf{A}^{n+2} \quad \text{given by} \quad (t, (x_0, \ldots, x_{n+1})) \mapsto (t^{a_0} x_0, \ldots, t^{a_{n+1}} x_{n+1}) \,.$$

The condition $\gcd(a_0, a_1, \ldots, a_{n+1}) = 1$ is equivalent to the action α being faithful. This $\mathbf{G}_\mathrm{m}$-action induces a grading on $\mathbf{C}[x_0, \ldots, x_{n+1}]$ where each variable x_i has degree a_i, for all $i \in \{0, 1, \ldots, n+1\}$. We define the weighted projective space with weights $a = (a_0, a_1, \ldots, a_{n+1})$, denoted by $\mathbf{P}_a^{n+1}$, as $\operatorname{Proj} \mathbf{C}[x_0, \ldots, x_{n+1}]$, see [Har77, Ch. II, Proposition 2.5] for details of the Proj construction.

In the sequel, we always consider the polynomial ring $\mathbf{C}[x_0, \ldots, x_{n+1}]$ with the grading given by $a \in \mathbf{Z}_{>0}^{n+2}$. The usual projective space is recovered from this construction by setting each $a_i = 1$. Remark that the vector a is not uniquely determined by $\mathbf{P}_a^{n+1}$ even up to reordering. For example, if $a = (1, 2, \ldots, 2)$, then $\mathbf{P}_a^{n+2}$ is isomorphic to the usual projective space $\mathbf{P}^{n+2}$. The standard references for weighted projective spaces are [Dol81, IF00].

The automorphism groups of weighted projective spaces are known and are a natural generalization of the case of the usual projective space, see [AA89, Ess24]. Recall that $\mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2})$ is the group of $\mathbf{G}_\mathrm{m}$-equivariant automorphisms of $\mathbf{A}^{n+2}$. Then

$$\mathrm{Aut}(\mathbf{P}_a^{n+1}) = \mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2})/H\,, \tag{1}$$

where H stands for the image of the $\mathbf{G}_\mathrm{m}$-action α inside $\mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2})$. Remark that if $a = (1, 1, \ldots, 1)$, then being $\mathbf{G}_\mathrm{m}$-equivariant stand for sending each variable to an element of degree 1, so $\mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2}) = \mathrm{GL}(n+2, \mathbf{C})$ and $\mathrm{Aut}(\mathbf{P}_a^{n+1}) = \mathrm{PGL}(n+2, \mathbf{C})$ in this case. Furthermore, we have the following generalization of the fact that finite abelian subgroups of $\mathrm{GL}(n+2, \mathbf{C})$ are diagonalizable.

Lemma 1.1 ([Ess24, Lemma 1.4])**.** *Let G be a finite abelian subgroup of $\mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2})$. Then G is conjugated to a diagonal automorphism of $\mathrm{Aut}_{\mathbf{G}_\mathrm{m}}(\mathbf{A}^{n+2})$, i.e., an automorphism sending each x_i to a scalar multiple of itself.*

Let now $a = (a_0, a_1, \ldots, a_{n+1}) \in \mathbf{Z}_{>0}^{n+2}$ be such that $\gcd(a_0, a_1, \ldots, a_{n+1}) = 1$. A polynomial

$$F \in \mathbf{C}[x_0, \ldots, x_{n+1}]\,,$$

is called homogeneous if it is homogeneous with respect to the grading induced in $\mathbf{C}[x_0, \ldots, x_{n+1}]$ by a. Let $F \in \mathbf{C}[x_0, \ldots, x_{n+1}]$ be an irreducible homogeneous polynomial. A hypersurface in $\mathbf{P}_a^{n+1}$ is the algebraic variety $X = V(F)$ defined as the zero locus of F in the weighted projective space $\mathbf{P}_a^{n+1}$, that is,

$$X = \left\{[x_0 : \cdots : x_{n+1}] \in \mathbf{P}_a^{n+1} \mid F(x) = 0\right\}\,.$$

The weighted projective space $\mathbf{P}_a^{n+1}$ is said to be well-formed if the corresponding $\mathbf{G}_\mathrm{m}$-action α on $\mathbf{A}^{n+2}$ has trivial stabilizers in codimension one. This occurs if and only if the greatest common divisor of every subset of $\{a_0, a_1, \ldots, a_{n+1}\}$ of size $n+1$ is equal to 1. By [IF00, Lemma 5.7], every weighted projective space is isomorphic to a well-formed one. Furthermore, a hypersurface $X \subset \mathbf{P}_a^{n+1}$ is said to be well-formed if $\mathbf{P}_a^{n+1}$ is well-formed and the intersection of X with the singular locus of $\mathbf{P}_a^{n+1}$ has codimension at least two in X.

Weighted projective spaces $\mathbf{P}_a^{n+1}$ that are not isomorphic to the usual projective space $\mathbf{P}^{n+1}$ are always singular. For this reason, a weaker notion of smoothness, known as quasi-smoothness, is introduced [Dan91, BC94]. Let $F \in \mathbf{C}[x_0, \ldots, x_{n+1}]$ be a homogeneous irreducible polynomial of degree d defining a hypersurface $X = V(F) \subset \mathbf{P}_a^{n+1}$. The affine cone C_X over X is the affine variety

$$C_X = \left\{(x_0, \ldots, x_{n+1}) \in \mathbf{A}^{n+2} \mid F(x) = 0\right\}\,.$$

We say that X is quasi-smooth if $C_X \setminus \{\bar{0}\}$ is smooth. Note that the notion of quasi-smoothness depends on the specific embedding $X \subset \mathbf{P}_a^{n+1}$. Quasi-smooth hypersurfaces of degree d in $\mathbf{P}_a^{n+1}$ exist only for certain combinations of weights and degree. Given a fixed choice of weights $a \in \mathbf{Z}_{>0}^{n+2}$, a characterization of the degrees d for which quasi-smooth hypersurfaces exist is provided in [IF00, Theorem 8.1]. We present here the formulation given in [Ess24].

Theorem 1.2 ([IF00, Theorem 8.1])**.** *Let $a \in \mathbf{Z}_{>0}^{n+2}$ and assume that $\mathbf{P}_a^{n+1}$ is well-formed. Then, there exists a quasi-smooth hypersurface X of degree d in the weighted projective space $\mathbf{P}_a^{n+1}$ if and only if one of the following conditions holds:*

- (i) *$a_i = d$ for some $i \in \{0, 1, \ldots, n+1\}$, or*
- (ii) *for each nonempty subset I of $\{0, 1, \ldots, n+1\}$, either*
  - *(a) d is contained in the subsemigroup of $\mathbf{Z}_{\geq 0}$ generated by the weights $\{a_i \mid i \in I\}$, or*
  - *(b) there are at least $|I|$ indices $j \notin I$ such that $d - a_j$ is contained in the subsemigroup of $\mathbf{Z}_{\geq 0}$ generated by the weights $\{a_i \mid i \in I\}$.*

Moreover, if (i) or (ii) holds, then the general hypersurface of degree d is quasi-smooth.

The following lemma is a version of Theorem 1.2 applied to singleton sets $I = \{i\}$. A direct proof is straightforward, see also [GAL13, Lemma 1.2].

Lemma 1.3 ([Ess24, Proposition 1.2])**.** *Let $X \subset \mathbf{P}_a^{n+1}$ be a quasi-smooth hypersurface, given by a polynomial $F \in \mathbf{C}[x_0, \dots, x_{n+1}]$ of degree d. Then for each $i \in \{0, 1, \dots, n+1\}$, there exists a monomial of degree d with nonzero coefficient in F having the form either x_i^k or $x_i^k x_j$, for some $j \neq i$.*

Let $X = V(F)$ be a hypersurface in $\mathbf{P}_a^{n+1}$, defined as the zero locus of a homogeneous polynomial $F \in \mathbf{C}[x_0, \dots, x_{n+1}]$ of degree d. The group of linear automorphisms, denoted by $\operatorname{Lin}(X)$, is the subgroup of $\operatorname{Aut}(X)$ consisting of automorphisms that extend to automorphisms of the ambient space $\mathbf{P}_a^{n+1}$, that is,

$$\operatorname{Lin}(X) = \{\varphi \in \operatorname{Aut}(\mathbf{P}_a^{n+1}) \mid \varphi(X) = X\}\,.$$

We refer to the elements of $\operatorname{Lin}(X)$ as linear automorphisms, in analogy with the classical case, even though $\operatorname{Aut}(\mathbf{P}_a^{n+1})$ is not necessarily the image of linear automorphisms of $\mathbf{A}^{n+2}$ under the quotient described in (1).

We say that X is a linear cone if $d = a_i$ for some $i \in \{0, 1, \dots, n+1\}$. In this case, the variable x_i appears as a linear summand in F, and thus X is isomorphic to the weighted projective space $\mathbf{P}_{a'}^{n}$, where $a' = (a_0, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1})$.

It is a classical result of Matsumura and Monsky [MM64] that for a hypersurface of degree d in the usual projective space $\mathbf{P}^{n+1}$ (i.e., with weights $a = (1, 1, \dots, 1)$), we have $\operatorname{Aut}(X) = \operatorname{Lin}(X)$ and the group $\operatorname{Aut}(X)$ is finite, except in the two exceptional cases $(n, d) = (1, 3)$ and $(n, d) = (2, 4)$. A natural generalization of this result to the setting of weighted projective spaces has been established in [Ess24, Theorem 2.1]. We now recall this result.

Theorem 1.4. *Let $X \subset \mathbf{P}_a^{n+1}$ and $X' \subset \mathbf{P}_{a'}^{n+1}$ be well-formed and quasi-smooth hypersurfaces of degrees d and d', respectively. Assume further that X is not a linear cone and let $\tau\colon X \to X'$ be an isomorphism.*

- (i) *If $n \geq 3$; or $n = 2$ and $a_0 + a_1 + a_2 + a_3 \neq d$, then $d = d'$, $a = a'$ up to reordering and τ is the restriction of an automorphism of $\mathbf{P}_a^{n+1}$. In particular, $\operatorname{Aut}(X) = \operatorname{Lin}(X)$.*
- (ii) *The group $\operatorname{Lin}(X)$ is finite if and only if $d > 2\max(a)$, or $d = 2\max(a)$ and the maximum is achieved only in one a_i.*

Unless otherwise stated, we assume throughout the paper that all hypersurfaces in weighted projective space are well-formed and not linear cones.

2. Automorphisms of prime power order of quasi-smooth hypersurfaces

In this section, we compute the powers of prime numbers that can occur as the order of an automorphism of a well-formed quasi-smooth hypersurface in a weighted projective space. We begin by introducing the notion of F-liftability. Let G be a subgroup of $\operatorname{Aut}(\mathbf{P}_a^{n+1})$. We say that G is liftable if there exists a subgroup $\widetilde{G} \subset \operatorname{Aut}_{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^{n+2})$ such that the canonical homomorphism

$$\pi\colon \operatorname{Aut}_{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^{n+2}) \to \operatorname{Aut}(\mathbf{P}_a^{n+1})$$

from (1) restricts to an isomorphism $\pi|_{\widetilde{G}}\colon \widetilde{G} \to G$. Given an element $\varphi \in G$, we say that $\widetilde{\varphi} \in \widetilde{G}$ is a lifting of φ if $\pi(\widetilde{\varphi}) = \varphi$.

Let now G be a subgroup of $\operatorname{Lin}(X)$, where X is a quasi-smooth hypersurface in $\mathbf{P}_a^{n+1}$ defined by a homogeneous polynomial $F \in \mathbf{C}[x_0, \dots, x_{n+1}]$ of degree d. We say that G is F-liftable if G is liftable and, for every $\widetilde{\varphi} \in \widetilde{G}$, we have $\widetilde{\varphi}^*(F) = F$ [OY19, GALM22]. Moreover, we say that an element $\varphi \in G$ is F-liftable if the cyclic group generated by φ is F-liftable.

We will use the following multi-index notation. Let F be a homogeneous polynomial in $\mathbf{C}[x_0, \dots, x_{n+1}]$ of degree d. Then we can write

$$F = \sum_i \lambda_i \cdot x^{m_i}\,,$$

where each $\lambda_i \in \mathbf{C}$, and $m_i = (m_{0,i}, m_{1,i}, \dots, m_{n+1,i}) \in \mathbf{Z}_{\geq 0}^{n+2}$ is a multi-index. Here, x^{m_i} denotes the monomial $x_0^{m_{0,i}} x_1^{m_{1,i}} \cdots x_{n+1}^{m_{n+1,i}}$. The condition that F is homogeneous of degree d with respect to the grading induced by

$a = (a_0, a_1, \ldots, a_{n+1})$ means that

$$\sum_{j=0}^{n+1} a_j m_{j,i} = d \quad \text{for each } i\,.$$

With these definitions and notations, we now state the following lemma.

Lemma 2.1. *Let n, d, p, r be positive integers, with p prime. Let $a \in \mathbf{Z}_{>0}^{n+2}$, and assume that p does not divide d. Further, assume that Theorem 1.4 (i) holds. If φ is an automorphism of order p^r of a quasi-smooth hypersurface $X \subset \mathbf{P}_a^{n+1}$, then φ is F-liftable.*

Proof. Assume that X is defined by a polynomial $F \in \mathbf{C}[x_0, \ldots, x_{n+1}]$ of degree d with respect to the grading given by $a = (a_0, a_1, \ldots, a_{n+1})$. Let $\widetilde{\varphi} \in \mathrm{Aut}_{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^{n+2})$ be a lifting of an automorphism $\varphi \in \mathrm{Aut}(\mathbf{P}_a^{n+1})$. By Lemma 1.1, we may and will assume that $\widetilde{\varphi}$ is diagonal. Then,

$$\widetilde{\varphi}^*(F) = \xi^h F\,,$$

where ξ is a primitive p^r-th root of unity and $h \in \mathbf{Z}$. Since p^r and d are relatively prime, we can choose $k \in \mathbf{Z}$ such that $kd \equiv -h \pmod{p^r}$. Now, consider the automorphism

$$\widetilde{\psi} \in \mathrm{Aut}_{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^{n+2}) \quad \text{defined by} \quad \widetilde{\psi}^*(x_i) = \xi^{ka_i}\widetilde{\varphi}^*(x_i)\,.$$

Now, assuming that $F = \sum_i \lambda_i \cdot x^{m_i}$ we obtain that

$$\begin{aligned}
\widetilde{\psi}^*(F) &= \sum_i \lambda_i \cdot \widetilde{\psi}^*(x^{m_i}) \\
&= \sum_i \lambda_i \cdot (\xi^{ka_0}\widetilde{\varphi}^*(x_0))^{m_{0,i}} \cdots (\xi^{ka_{n+1}}\widetilde{\varphi}^*(x_{n+1}))^{m_{n+1,i}} \\
&= \xi^{kd} \sum_i \lambda_i \cdot \widetilde{\varphi}^*(x_0)^{m_{0,i}} \cdots \widetilde{\varphi}^*(x_{n+1})^{m_{n+1,i}} \\
&= \xi^{kd}\widetilde{\varphi}^*(F) = \xi^{kd+h} F = F
\end{aligned}$$

Note that $\widetilde{\psi}$ and $\widetilde{\varphi}$ induce the same automorphism in $\mathbf{P}_a^{n+1}$ by (1). We conclude that φ is F-liftable. □

We now establish our main technical result, which generalizes [GAL13, Theorem 1.3].

Proposition 2.2. *Let n, d, p, r be positive integers, with p prime and $d \geq 3$. Let $a \in \mathbf{Z}_{>0}^{n+2}$, and assume that p divides neither d nor $d - a_i$, for any $i \in \{0, 1, \ldots, n+1\}$. Further, assume that Theorem 1.4 (i) holds. If p^r is the order of an automorphism of a quasi-smooth hypersurface in $\mathbf{P}_a^{n+1}$, then, after possibly reordering the weights $a = (a_0, a_1, \ldots, a_{n+1})$, there exists $\ell \in \{1, 2, \ldots, n+1\}$ such that the following conditions hold:*

(i) $d \equiv a_0 \pmod{a_\ell}$, and $d \equiv a_{i+1} \pmod{a_i}$ for all $0 \leq i < \ell$;

(ii) $(-1)^{\ell+1} \prod_{t=0}^{\ell} \left(\frac{d - a_t}{a_t}\right) \equiv 1 \pmod{p^r}$.

Proof. Let X be a quasi-smooth hypersurface in $\mathbf{P}_a^{n+1}$, defined by a polynomial $F \in \mathbf{C}[x_0, \ldots, x_{n+1}]$ of degree d with respect to the grading given by $a = (a_0, a_1, \ldots, a_{n+1})$. To prove the proposition, assume that X admits an automorphism φ of order p^r. By Lemma 1.1, we may and will assume that φ is diagonal. Moreover, by Lemma 2.1, we may also assume that φ is F-liftable. Let $\widetilde{\varphi}$ be a lifting of φ such that $\widetilde{\varphi}^*(F) = F$.

Letting ξ be a primitive p^r-th root of unity, we have

$$\widetilde{\varphi}(x_0, \ldots, x_{n+1}) = (\xi^{\sigma_0}x_0, \ldots, \xi^{\sigma_{n+1}}x_{n+1}) \quad \text{with} \quad 0 \leq \sigma_i < p^r\,.$$

By Lemma 1.3, the homogeneous polynomial F contains a monomial of the form $x_i^{m_i}$ or $x_i^{m_i}x_j$ for each $i \in \{0, 1, \ldots, n+1\}$ and some $j \neq i$. Assume that F contains a monomial of the form $x_i^{m_i}$, for some $i \in \{0, 1, \ldots, n+1\}$. Then

$$a_i m_i = d \quad \text{and} \quad \sigma_i m_i \equiv 0 \pmod{p^r}\,.$$

This implies that p^r divides $\sigma_i m_i$ so that p divides m_i, and hence p divides d, contradicting our assumption. We conclude that, up to reordering the weights $a = (a_0, a_1, \ldots, a_{n+1})$, we may assume that $\sigma_0 \neq 0$ and that the monomials

$$x_0^{m_0}x_1,\ x_1^{m_1}x_2,\ \ldots,\ x_\ell^{m_\ell}x_0$$

belong to F, for some $1 \leq \ell \leq n+1$. Since each monomial has degree d, we obtain the following equalities:

$$a_0 + m_\ell a_\ell = d \quad \text{and} \quad a_i m_i + a_{i+1} = d \quad \text{for all } 0 \leq i < \ell\,. \tag{2}$$

These relations yield condition (i) in the proposition.

The monomial $x_0^{m_0}x_1 \in F$ is invariant by the automorphism $\widetilde{\varphi}^*$, so $\sigma_0 m_0 + \sigma_1 \equiv 0 \pmod{p^r}$. By (2), we have that

$$\sigma_1 \equiv -\left(\frac{d-a_1}{a_0}\right)\sigma_0 \pmod{p^r}\,, \tag{3}$$

and since p does not divide $d - a_1$, we conclude that $\sigma_1 \not\equiv 0 \pmod{p^r}$.

Applying the above argument to the monomial $x_1^{m_1}x_2 \in F$, we obtain that $\sigma_1 m_1 + \sigma_2 \equiv 0 \pmod{p^r}$ and by (2) and (3), we have that $\sigma_2 \not\equiv 0 \pmod{p^r}$ and

$$\sigma_2 \equiv (-1)^2\,\frac{(d-a_2)(d-a_1)}{a_1 a_0}\,\sigma_0 \pmod{p^r}\,.$$

Iterating this process for all $j \in \{2, 3, \ldots, \ell-1\}$, we obtain that

$$\sigma_{j+1} \equiv (-1)^{j+1}\,\frac{\prod_{t=1}^{j+1}(d-a_t)}{\prod_{t=0}^{j} a_t}\,\sigma_0 \equiv (-1)^{j+1}\prod_{t=1}^{j+1}\left(\frac{d-a_t}{a_{t-1}}\right)\sigma_0 \pmod{p^r}\,. \tag{4}$$

Finally, the monomial $x_\ell^{m_\ell}x_0 \in F$ is also invariant by the automorphism $\widetilde{\varphi}^*$, so $\sigma_\ell m_\ell + \sigma_0 \equiv 0 \pmod{p^r}$. By (2) and (4), we conclude that

$$1 \equiv (-1)^{\ell+1}\prod_{t=0}^{\ell}\left(\frac{d-a_t}{a_t}\right) \pmod{p^r}\,.$$

This proves (ii) and concludes the proof. ☐

Let φ be an automorphism of order p of a quasi-smooth hypersurface $X \subset \mathbf{P}_a^{n+1}$. As in the proof of Proposition 2.2, we may and will assume that φ is induced by a diagonal automorphism of the affine space $\mathbf{A}^{n+2}$. Hence,

$$\widetilde{\varphi}\colon \mathbf{A}^{n+2} \to \mathbf{A}^{n+2} \quad \text{is given by} \quad (x_0, \ldots, x_{n+1}) \mapsto (\xi^{\sigma_0}x_0, \ldots, \xi^{\sigma_{n+1}}x_{n+1})\,,$$

where ξ is a primitive p^r-th root of unity and $\sigma_i \in \mathbf{Z}$. Since $\xi^{p^r} = 1$, the integers σ_i may be considered modulo p^r, i.e., as elements of $\mathbf{Z}/p^r\mathbf{Z}$. The vector $(\sigma_0, \sigma_1, \ldots, \sigma_{n+1}) \in (\mathbf{Z}/p^r\mathbf{Z})^{n+2}$ is called the signature of φ.

Remark 2.3. Let φ be an automorphism of order p^r of a quasi-smooth hypersurface $X = V(F)$, with $\gcd(p, d) = 1$ and $\gcd(p, d - a_i) = 1$ for all $i \in \{0, 1, \ldots, n+1\}$. Let σ_0 be as in the proof of Proposition 2.2. By (4), $\gcd(p, \sigma_0) = 1$, since otherwise the order of φ is less than p^r. Hence, there exists $k \in \{1, 2, \ldots, p-1\}$ such that $k\sigma_0 \equiv 1 \pmod{p^r}$. Hence, we can replace φ by φ^k, which generates the same cyclic subgroup of $\mathrm{Aut}(X)$. Therefore, we may and will assume that $\sigma_0 = 1$.

With this assumption, it follows from the proof of Proposition 2.2 that the automorphism φ has signature

$$\sigma = \left(1,\ -\left(\frac{d-a_1}{a_0}\right),\ (-1)^2\frac{(d-a_2)(d-a_1)}{a_1a_0},\ \ldots,\ (-1)^\ell\prod_{t=1}^{\ell}\left(\frac{d-a_t}{a_{t-1}}\right),\ \underbrace{*,\ldots,*}_{(n+2-\ell)\text{-times}}\right) \in (\mathbf{Z}/p^r\mathbf{Z})^{n+2}\,,$$

where the entries denoted by $*$ indicate unknown values.

Example 2.4. In this example we show that the converse of the Proposition 2.2 does not hold. Lets consider the weights $a = (3, 7, 2, 4, 5)$ and let $\mathbf{P}_a^{n+1} = \mathbf{P}_a^4$ be the corresponding weighted projective space. We also fix the degree $d = 37$. We have that the conditions (i) and (ii) in Proposition 2.2 hold with $p = 23$ and $r = 1$. Indeed, taking $\ell = 2$ yields (i) and (ii) follows since

$$(-1)^3\left(\frac{37-3}{3}\right)\left(\frac{37-7}{7}\right)\left(\frac{37-2}{2}\right) = -23\cdot 37 + 1 \equiv 1 \pmod{23}.$$

Nevertheless, we will prove that there is no quasi-smooth hypersurface in the weighted projective space with weight $a = (3, 7, 2, 4, 5)$ that admits an automorphism φ of order $p = 23$.

Assume such φ exists and let $\widetilde{\varphi}$ be a lifting that leaves F invariant. Such lifting exists by Lemma 2.1. By Remark 2.3, the polynomial $F \in \mathbf{C}[x_0, \ldots, x_{n+1}]$ defining a quasi-smooth hypersurface in the weighted projective space $\mathbf{P}_a^4$ need to contain the monomials $x_0^{10}x_1$, $x_1^5x_2$, and $x_2^{17}x_0$ with non-zero coefficient and the signature of the automorphism φ is

$$\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, -10, 10\cdot 5, *, *) = (1, 13, 4, *, *) \in (\mathbf{Z}/23\mathbf{Z})^5 .$$

By Lemma 1.3, we have that $x_3^8x_4$ and at least one of the monomials $x_4^6x_1$ and $x_4^7x_2$ have to appear in F with non-zero coefficient. A straightforward application of the Jacobian Criterion, shows that indeed $x_4^6x_1$ and $x_4^7x_2$ both have to appear with non-zero coefficient. Finally, these two monomial provide contradictory conditions for the weight of x_4. Indeed, the fact that F is invariant by $\widetilde{\varphi}^*$ yields

$$6\sigma_4 + \sigma_1 \equiv 0 \pmod{23} \quad \text{and} \quad 7\sigma_4 + \sigma_2 \equiv 0 \pmod{23}.$$

This provides a contradiction since the first equation yields $\sigma_4 = 17$ while the second one yields $\sigma_4 = 6$.

In the next proposition, we provide a partial converse to Proposition 2.2.

Proposition 2.5. *Let n, d, p, r be positive integers, with p prime and $d \geq 3$. Let $a \in \mathbf{Z}_{>0}^{n+2}$ and assume that p divides neither d nor $d - a_i$, for any $i \in \{0, 1, \ldots, n+1\}$. Further, assume that Theorem 1.4 (i) holds. If, after possibly reordering the weights $a = (a_0, a_1, \ldots, a_{n+1})$, the following conditions are satisfied:*

- (i) *$d \equiv a_0 \pmod{a_\ell}$, and $d \equiv a_{i+1} \pmod{a_i}$ for all $0 \leq i < \ell$;*
- (ii) *$(-1)^{\ell+1}\prod_{t=0}^{\ell}\left(\frac{d-a_t}{a_t}\right) \equiv 1 \pmod{p^r}$; and*
- (iii) *There exists a quasi-smooth hypersurface $X = V(F)$ of degree d in $\mathbf{P}_a^{n+1}$ with $F = F_1 + F_2$ where $F_1 \in \mathbf{C}[x_0, \ldots, x_\ell]$ and $F_2 \in \mathbf{C}[x_{\ell+1}, \ldots, x_{n+1}]$;*

then p^r is the order of an automorphism of a quasi-smooth hypersurface of $\mathbf{P}_a^{n+1}$

Proof. To prove the proposition, it is enough to provide a quasi-smooth hypersurface of dimension n and degree d in $\mathbf{P}_a^{n+1}$ admitting an automorphism of order p^r. By (i) in the proposition, up to reordering the set of weights $a = (a_0, a_1, \ldots, a_{n+1})$, there exists an index $1 \leq \ell \leq n+1$ such that $a_\ell m_\ell + a_0 = d$ and $a_i m_i + a_{i+1} = d$, for all $i \in \{0, 1, \ldots, \ell - 1\}$, where every $m_i \in \mathbf{Z}_{>0}$. This ensures that the polynomial

$$F_1' = \sum_{j=0}^{\ell-1} x_i^{m_i}x_{i+1} + x_\ell^{m_\ell}x_0,$$

is homogeneous of degree d in $\mathbf{C}[x_0, \ldots, x_\ell]$ with weights $(a_0, a_1, \ldots, a_\ell)$. Moreover, we will show in Proposition 3.4 below that the hypersurface defined by F_1' in $\mathbf{P}_a^\ell$ is quasi-smooth. By (iii) in the proposition, we know that there exists F_1 and F_2 such that $F_1 + F_2$ defines a quasi-smooth hypersurface in $\mathbf{P}_a^{n+1}$. The Jacobian Criterion implies that $F_1 + F_2$ defines a quasi-smooth hypersurface if and only if F_1 and F_2 also define quasi-smooth hypersurfaces. Hence, we can assume without loss of generality that $F_1 = F_1'$.

Finally, by (ii) in the proposition, we have that

$$(-1)^{\ell+1}\prod_{t=0}^{\ell}\left(\frac{d-a_t}{a_t}\right) \equiv 1 \pmod{p^r}.$$

By this last relation, we have that $F = F_1 + F_2$ is invariant by the diagonal automorphism with signature

$$\sigma = \left(1, -\left(\frac{d-a_1}{a_0}\right), \frac{(d-a_2)(d-a_1)}{a_1 a_0}, \dots, (-1)^{\ell} \prod_{t=1}^{\ell} \left(\frac{d-a_t}{a_{t-1}}\right), \underbrace{0, \dots, 0}_{(n+2-\ell)\text{- times}}\right) \in (\mathbf{Z}/p^r\mathbf{Z})^{n+2}.$$

Hence, the corresponding quasi-smooth hypersurface $X = V(F)$ in $\mathbf{P}_a^{n+1}$ is invariant under the automorphism φ of order p^r. This concludes the proof. □

The most frequently encountered case of hypersurfaces in weighted projective space in geometric applications is when each a_i divides d. In this case, we provide in Theorem 2.6 a complete criterion for determining the prime numbers that appear as the order of an automorphism of a smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$.

Theorem 2.6 includes the classical projective space, where each $a_i = 1$, which was proven in [GAL13, Proposition 2.2] and can be regarded as its generalization. Moreover, in [ST24], the authors study the K-stability of quasi-smooth weighted Fano hypersurfaces $X \in \mathbf{P}_a^{n+1}$ of degree d, and also characterize the finite automorphism group of quasi-smooth Fano weighted complete intersections, under the assumption that each weight a_i divides d, so our Theorem 2.6 applies. In addition, the classification of Fano threefolds containing a smooth rational surface with ample normal bundle, as presented in [CF93], also falls under the hypotheses of Theorem 2.6, as we show below in Example 2.9.

Theorem 2.6. *Let n, d, p be positive integers, with p prime and $d \geq 3$. Let $a \in \mathbf{Z}_{>0}^{n+2}$ and assume that Theorem 1.4 (i) holds. If a_i divides d for all $i \in \{0, 1, \dots, n+1\}$, then p is the order of an automorphism of a well formed quasi-smooth hypersurface X of $\mathbf{P}_a^{n+1}$ if and only if one of the following conditions hold:*

- (a) *p divides d; or*
- (b) *$a_i p$ divides $d - a_j$, for some $i, j \in \{0, 1, \dots, n+1\}$ with $i \neq j$; or*
- (c) *after possibly reordering the weights $a = (a_0, a_1, \dots, a_{n+1})$, there exists ℓ with $1 \leq \ell \leq n+1$ such that $a_0 = a_1 = \cdots = a_\ell$ and*

$$\left(1 - \frac{d}{a_0}\right)^{\ell+1} \equiv 1 \pmod{p}.$$

Proof. To prove the "only if" part of the theorem, assume that X is a quasi-smooth hypersurface that admits an automorphism φ of order p prime. If p divides d, then we are in case (a). If there exists i, j with $i \neq j$ such that $a_i p$ divides $d - a_j$, then we are in case (b).

Assume now that neither of this two conditions happen. Then, in particular, p divides neither d nor $d - a_i$for $i \in \{0, 1, \dots, n+1\}$, so we are in the setup of Proposition 2.2. By Proposition 2.2, there exists an index $\ell \in \{1, 2, \dots, n+1\}$ such that the following conditions hold:

- (i) $d \equiv a_0 \pmod{a_\ell}$, and $d \equiv a_{i+1} \pmod{a_i}$ for all $0 \leq i < \ell$; and
- (ii) $(-1)^{\ell+1} \prod_{t=0}^{\ell} \left(\frac{d-a_t}{a_t}\right) \equiv 1 \pmod{p}$.

By (i) and the hypothesis of the theorem, we have that exists $k_i, m_i \in \mathbf{Z}_{>0}$ such that

$$a_\ell m_\ell + a_0 = d = k_\ell a_\ell \quad \text{and} \quad a_i m_i + a_{i+1} = d = k_i a_i, \quad \text{for all } 0 \leq i < \ell.$$

We conclude that a_ℓ divides a_0 and a_i divides a_{i+1} for all $0 \leq i < \ell$. This yields $a_0 = a_1 = \cdots = a_\ell$. And now (c) follows directly from (ii).

To prove the "if" part of the theorem, it is enough to provide a quasi-smooth hypersurface $X = V(F)$ of dimension n and degree d in $\mathbf{P}_a^{n+1}$ admitting an automorphism of order p in each of the cases.

Assume first that (a) is fulfilled, i.e., p divides d. Letting $m_k = \frac{d}{a_k}$ for all $k \in \{0, 1, \dots, n+1\}$ we let $X = V(F)$, where

$$F = x_0^{m_0} + x_1^{m_1} \cdots + x_n^{m_n} + x_{n+1}^{m_{n+1}}.$$

A direct computation shows that X is quasi-smooth. Since $\gcd(a_0, a_1, \ldots, a_{n+1}) = 1$, we have that $\gcd(a_i, p) = 1$ for some $i \in \{0, 1, \ldots, n+1\}$. Since p divides d and $\gcd(a_i, p) = 1$ we have that p divides m_i. This yields that X admits the automorphism of order p whose signature is

$$\sigma = (1, 1, \ldots, 1, \underbrace{\xi_p}_{i\text{-th place}}, 1, \ldots, 1)$$

where ξ_p is a primitive p-th root of unity, proving the theorem in this case.

Assume now that (b) is fulfilled, i.e., $a_i p$ divides $d - a_j$, for some $i, j \in \{0, 1, \ldots, n+1\}$ with $i \neq j$. Letting $m_k = \frac{d}{a_k}$ for all $k \neq i$ and $m_i = \frac{d-a_j}{a_i}$, we let $X = V(F)$, where

$$F = x_i^{m_i} x_j + \sum_{k \neq i} x_k^{m_k}\,.$$

A direct computation shows that X is quasi-smooth. Since $a_i p$ divides $d - a_j$, we have that p divides m_i. This yields that X admits the automorphism of order p whose signature is

$$\sigma = (1, 1, \ldots, 1, \underbrace{\xi_p}_{i\text{-th place}}, 1, \ldots, 1)$$

where ξ_p is a primitive p-th root of unity, proving the theorem in this case.

Finally, assume that (c) is fulfilled. In this case, Proposition 2.5 (iii) holds for every reordering of the weights and every ℓ. Now, the existence of a quasi-smooth hypersurface $X = V(F)$ in $\mathbf{P}_a^{n+1}$ admitting an automorphism of order p is guaranteed by Proposition 2.5. Moreover, the explicit polynomial F defining X is provided in the proof of Proposition 2.5. This proves the theorem in this case and concludes the proof. $\square$

Remark 2.7. In Theorem 2.6, if the weighted projective space $\mathbf{P}_a^{n+1}$ is different from the usual projective space, i.e., if $a \neq (1, 1, \ldots, 1)$, then ℓ in (c) has to be less or equal than $n-1$ since otherwise $\mathbf{P}_a^{n+1}$ is not well-formed.

Corollary 2.8. *Let n, d, p be positive integers with p prime and $d \geq 3$. Let $a \in \mathbf{Z}_{>0}^{n+2}$ be such that a_i divides d for all $i \in \{0, 1, \ldots, n+1\}$. Assume that Theorem 1.4 (i) holds. If p is the order of an automorphism of a quasi-smooth hypersurface X of dimension n and degree d in $\mathbf{P}_a^{n+1}$, then*

$$p \leq \max\left\{ d, \left(\frac{d}{a_i} - 1\right)^{n_i - 1}, \textit{for all } i \in \{0, 1, \ldots, n+1\} \right\},$$

where n_i be the number of times that the weight a_i appears in a.

Proof. We will prove the corollary by contradiction. Assume that

$$p > \max\left\{ d, \left(\frac{d}{a_i} - 1\right)^{n_i - 1}, \text{ for all } i \in \{0, 1, \ldots, n+1\} \right\}.$$

Hence,

$$p > d, \quad \text{and} \quad p > \left(\frac{d}{a_i} - 1\right)^{n_i - 1}, \text{ for all } i \in \{0, 1, \ldots, n+1\} \tag{5}$$

Since p is the order of an automorphism of a of a quasi-smooth hypersurface, by Theorem 2.6, we have that one of following conditions hold:

- (a) p divides d; or
- (b) $a_i p$ divides $d - a_j$, for some $i, j \in \{0, 1, \ldots, n+1\}$ with $i \neq j$; or
- (c) after possibly reordering the weights $a = (a_0, a_1, \ldots, a_{n+1})$, there exists $\ell \in \{1, 2, \ldots, n+1\}$ such that $a_0 = a_1 = \cdots = a_\ell$ and

$$\left(1 - \frac{d}{a_0}\right)^{\ell+1} \equiv 1 \pmod{p}.$$

If (a) or (b) hold, then $p \leq d$ which yields a contradiction. Assume now that (c) holds. Since $a_0 = a_1 = \cdots = a_\ell$, we have that $\ell + 1 \leq n_0$. Then

$$\left(1 - \frac{d}{a_0}\right)^{\ell+1} - 1 = kp, \quad \text{for some } k \in \mathbf{Z}, \text{ and so}$$

$$\left(\frac{d}{a_0} - 1\right)^{\ell+1} + (-1)^{\ell} = kp, \quad \text{for some } k \in \mathbf{Z}_{>0}. \tag{6}$$

By (5), we have that $p > \left(\frac{d}{a_0} - 1\right)^{n_0-1}$. This yields $\ell + 1 = n_0 - 1$ or $\ell + 1 = n_0$. Assume first that $\ell + 1 = n_0 - 1$. Then by (6), we have that the this is only possible if $k = 1$ yielding

$$\left(\frac{d}{a_0} - 1\right)^{n_0-1} + (-1)^{n_0-2} = p\,.$$

Assume now that $\ell + 1 = n_0$. Then by (6), we have that the this is only possible if

$$\left(\frac{d}{a_0} - 1\right)^{n_0} + (-1)^{n_0-1} = kp, \quad \text{for some } k \in \left\{1, 2, \ldots, \tfrac{d}{a_0} - 1\right\}.$$

In both case, we conclude by the binomial expansion that $\frac{d}{a_0} \geq 2$ divides p providing a contradiction. □

As an application of Theorem 2.6, we provide the following example.

Example 2.9. For simplicity, in this example we denote the weighted projective space $\mathbf{P}^{n+1}_a$ simply by $\mathbf{P}(a)$. The classification of Fano threefolds containing a smooth rational surface with ample normal bundle, as presented in [CF93], includes four families of hypersurfaces: a smooth cubic hypersurface in $\mathbf{P}^4 = \mathbf{P}(1,1,1,1,1)$, a quartic hypersurface in $\mathbf{P}(1,1,1,1,2)$, a sextic hypersurface in $\mathbf{P}(1,1,1,2,3)$, and a sextic hypersurface in $\mathbf{P}(1,1,2,2,3)$; see also [Pro25].

It follows from Theorem 2.6 that smooth cubic hypersurfaces in $\mathbf{P}^4$ may admit automorphisms of order $p = 2$, 3, 5 and 11; quartic hypersurface in $\mathbf{P}(1,1,1,1,2)$ may have automorphisms of order $p = 2$, 3, 5 and 7; sextic hypersurface in $\mathbf{P}(1,1,1,2,3)$ may admit automorphisms of order $p = 2$, 3, 5 and 7; and sextic hypersurface in $\mathbf{P}(1,1,2,2,3)$ may have automorphisms of order $p = 2$, 3 and 5. We provide the computation of the case of the sextic hypersurface in $\mathbf{P}(1,1,1,2,3)$ as an example of the computations. By Theorem 2.6 (i) and (ii), provide that there are quasi-smooth hypersurfaces with automorphism of orders 2, 3 and 5. Now, to apply Theorem 2.6 (iii), we have that $\ell = 1$ or $\ell = 2$. Then the possible prime orders different from 2, 3 and 5 satisfy

$$(1-6)^2 = 2^3 \cdot 3 + 1 \equiv 1 \pmod{p}, \quad \text{and} \quad (1-6)^3 = -2 \cdot 3^2 \cdot 7 + 1 \equiv 1 \pmod{p}.$$

Then, the sextic hypersurface in $\mathbf{P}(1,1,1,2,3)$ admits automorphisms of order primer $p = 2$, 3, 5 and 7.

3. Automorphism of maximal prime order and Klein hypersurfaces

In Remark 2.7, we showed that the maximal prime numbers that can appear as the order of an automorphism of a quasi-smooth hypersurface in a weighted projective space are relatively small compared to the classical case of usual projective space, under the assumption that every weight a_i divides d. In this section, we investigate the opposite case, where every weight a_i is relatively prime to d.

In this context, similarly to the situation in [GAL13], Klein hypersurfaces naturally arise as the varieties admitting automorphisms of the largest possible prime order. We begin by proving the following corollary, which provides a bound in this setting.

Corollary 3.1. *Let n, d, p be positive integers, with $p > d$ a prime number and and $d \geq 3$. Let $a \in \mathbf{Z}^{n+2}_{>0}$, and assume that $\gcd(a_i, d) = 1$ for all $i \in \{0, 1, \ldots, n+1\}$. Further, assume that the condition in Theorem 1.4 (i)*

holds. If p is the order of an automorphism of a quasi-smooth hypersurface of dimension n and degree d in $\mathbf{P}_a^{n+1}$, then

$$p < \left(\frac{\max(a)}{d - \max(a)}\right) \prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t}\right),$$

where $\max(a)$ denotes the maximum of the weights $a_0, a_1, \ldots, a_{n+1}$.

Proof. We prove the statement by contradiction. Assume that $p > \left(\frac{\max(a)}{d-\max(a)}\right) \prod_{t=0}^{n+1} \left(\frac{d-a_t}{a_t}\right)$. Up to reordering the weight, we may and will assume that $\max(a) = a_{n+1}$ so that

$$p > \left(\frac{\max(a)}{d - \max(a)}\right) \prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t}\right) = \prod_{t=0}^{n} \left(\frac{d - a_t}{a_t}\right). \tag{7}$$

Since $p > d$ is the order of an automorphism of a of a quasi-smooth hypersurface, by Proposition 2.2, we have that there exists an index $\ell \in \{1, 2, \ldots, n+1\}$ such that the following conditions hold:

(i) $d \equiv a_0 \pmod{a_\ell}$, and $d \equiv a_{i+1} \pmod{a_i}$ for $0 \leq i < \ell$; and

(ii) $(-1)^{\ell+1} \prod_{t=0}^{\ell} \left(\frac{d - a_t}{a_t}\right) \equiv 1 \pmod{p}$.

Now, (ii) yields

$$\prod_{t=0}^{\ell} \left(\frac{d}{a_t} - 1\right) + (-1)^{\ell} = kp, \quad \text{for some } k \in \mathbf{Z}_{>0}. \tag{8}$$

By (7), we have that $\ell = n$ or $\ell = n+1$. Assume first that $\ell = n$. Then, by (8), we have that this is only possible if $k = 1$ yielding

$$\prod_{t=0}^{n} \left(\frac{d}{a_t} - 1\right) + (-1)^{n} = p.$$

Since $\gcd(a_i, d) = 1$ for every i, we can take this last equality modulo d to conclude that d divides p which is a contradiction.

Assume now that $\ell = n+1$. Then by (8), we have that this is only possible if

$$\prod_{t=0}^{n+1} \left(\frac{d}{a_t} - 1\right) + (-1)^{n+1} = kp, \quad \text{for some } k \in \left\{1, 2, \ldots, \tfrac{d}{a_{n+1}} - 1\right\}.$$

Again, since $\gcd(a_i, d) = 1$ for every i, we can take this last equality modulo d to conclude that d divides kp. Since $k < d$, we conclude that some divisor $d_0 > 1$ of d divides p which is again a contradiction. ☐

We now introduce the natural notion of a Klein hypersurface in the context of weighted projective spaces.

Definition 3.2. Let n, d be positive integers, and let $a \in \mathbf{Z}_{>0}^{n+2}$. Let X be a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$. We say that X is a Klein hypersurface in $\mathbf{P}_a^{n+1}$ if X is isomorphic to $V(K)$, where K is a homogeneous polynomial of the form

$$K = x_0^{m_0} x_1 + x_1^{m_1} x_2 + \ldots + x_n^{m_n} x_{n+1} + x_{n+1}^{m_{n+1}} x_0.$$

Remark 3.3. The Klein Hypersurface does not exists for every choice of d and a. Indeed, a straightforward verification shows that there is no Klein Hypersurface of degree 4 in $\mathbf{P}_a^3$ when $a = (1, 1, 1, 2)$.

In the following proposition we show that a Klein Hypersurface is almost always quasi-smooth. See also [GAL13, Example 3.5].

Proposition 3.4. *Let n, d be positive integers with $d \geq 2$. Let $a \in \mathbf{Z}_{>0}^{n+2}$. Assume that $X = V(K)$ is a Klein Hypersurface in $\mathbf{P}_a^{n+1}$. Then X is not quasi-smooth if and only if $a = (1, 1, \ldots, 1)$, $d = 2$, and $n \equiv 2 \pmod{4}$.*

Proof. Assume that $\alpha = [\alpha_0 : \alpha_1 : \cdots : \alpha_{n+1}] \in \mathbf{P}_a^{n+1}$ is a singular point of X. Then

$$K(\alpha) = 0 \quad \text{and} \quad \frac{\partial K}{\partial x_i}(\alpha) = 0, \quad \text{for all } i \in \{0, 1, \dots, n+1\}.$$

This yields

$$\begin{aligned} \frac{\partial K}{\partial x_0}(\alpha) &= \alpha_{n+1}^{m_{n+1}} + m_0 \alpha_0^{m_0 - 1} \alpha_1 = 0, \\ \frac{\partial K}{\partial x_i}(\alpha) &= \alpha_{i-1}^{m_{i-1}} + m_i \alpha_i^{m_i - 1} \alpha_{i+1} = 0, \text{ for all } i \in \{1, 2, \dots, n\}, \text{ and} \\ \frac{\partial K}{\partial x_{n+1}}(\alpha) &= \alpha_n^{m_n} + m_{n+1} \alpha_{n+1}^{m_{n+1} - 1} \alpha_0 = 0\,. \end{aligned}$$

Remark that if any $\alpha_j = 0$, the identities above yield that $\alpha_i = 0$, for all $i \in \{0, 1, \dots, n+1\}$. Hence, we may and will assume that $\alpha_i \neq 0$, for all $i \in \{0, 1, \dots, n+1\}$. Multiplying each of these identities by α_i, we obtain

$$\begin{aligned} \alpha_{n+1}^{m_{n+1}} \alpha_0 &= -m_0 \alpha_0^{m_0} \alpha_1, \\ \alpha_{i-1}^{m_{i-1}} \alpha_i &= -m_i \alpha_i^{m_i} \alpha_{i+1}, \text{ for all } i \in \{1, 2, \dots, n\}, \text{ and} \\ \alpha_n^{m_n} \alpha_{n+1} &= -m_{n+1} \alpha_{n+1}^{m_{n+1}} \alpha_0\,. \end{aligned}$$

Then, we obtain that

$$\alpha_i^{m_i} \alpha_{i+1} = (-1)^{n+1-i} \alpha_{n+1} \alpha_0 \left(\prod_{j=i+1}^{n+1} m_j \right), \quad \text{for all } i \in \{0, 1, \dots, n\}.$$

Replacing these identities in $K(\alpha)$, we obtain

$$K(\alpha) = R \cdot \alpha_{n+1} \alpha_0 \quad \text{where} \quad R = 1 + \sum_{i=1}^{n+1} \left[(-1)^{n-i} \left(\prod_{j=i}^{n+1} m_j \right) \right].$$

Now, in this last equality we have $R = 0$ if and only if $a = (1, 1, \dots, 1)$, $d = 2$, and n is even. If $a = (1, 1, \dots, 1)$ and $d = 2$ then K is a quadratic polynomial. A routine computation shows that in this case $X = V(K)$ is not smooth if and only if $n \equiv 2 \pmod 4$ proving the proposition. ☐

Let n, d, p be positive integers with $p > d$ prime. Let $a \in \mathbf{Z}_{>0}^{n+2}$ and assume that $\gcd(a_i, d) = 1$ for all $i \in \{0, 1, \dots, n+1\}$. Assume further that the condition in Theorem 1.4 (i) is fulfilled. Let X be a quasi-smooth hypersurface of degree d in $\mathbf{P}_a^{n+1}$ and assume that p is the order of an automorphism of X. By Proposition 2.2, we have that

$$(-1)^{\ell+1} \prod_{t=0}^{\ell} \left(\frac{d - a_t}{a_t} \right) \equiv 1 \pmod p\,.$$

The largest this prime number can be is

$$\prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t} \right) + (-1)^{n+1}$$

but under the hypothesis that $\gcd(a_i, d) = 1$ we have that this number is divisible by d. Hence, the largest prime that can be the order of an automorphism quasi-smooth hypersurface of $\mathbf{P}_a^{n+1}$ is

$$p = \frac{1}{d} \left[\prod_{t=0}^{n+1} \left(\frac{d - a_t}{a_t} \right) + (-1)^{n+1} \right].$$

In this context, we will prove the following theorem that is the main result of this section.

Theorem 3.5. *Let n, d be positive integers with $d \geq 3$. Let $a \in \mathbf{Z}_{>0}^{n+2}$. Assume further that the condition in Theorem 1.4 (i) is fulfilled. Let*

$$p = \frac{1}{d}\left[\prod_{t=0}^{n+1}\left(\frac{d-a_t}{a_t}\right) + (-1)^{n+1}\right],$$

and assume that $p > d$ and that p is prime. If a quasi-smooth hypersurface $X = V(F)$ of dimension n and degree d admits an automorphism φ of order p then X is isomorphic to the Klein hypersurface.

Proof. Assume that $X = V(F)$ is a quasi-smooth hypersurface of dimension n and degree d, where F is a homogeneous polynomial of degree d with respect to the grading given by $a \in \mathbf{Z}_{>0}^{n+2}$. Assume further that X admits an automorphism φ of order $p > d$ prime, where

$$p = \frac{1}{d}\left[\prod_{t=0}^{n+1}\left(\frac{d-a_t}{a_t}\right) + (-1)^{n+1}\right]. \tag{9}$$

Let S be the subspace of $\mathbf{C}[x_0, \ldots, x_{n+1}]$ composed of homogeneous polynomials of degree d with respect to the grading given by $a \in \mathbf{Z}_{>0}^{n+2}$. We also let $\widetilde{\varphi}^* \colon S \to S$ be action of $\widetilde{\varphi}^*$ on S. Let $\mathscr{E} \subset S$ be the eigenspace associated to the eigenvalue 1 of the linear automorphism $\widetilde{\varphi}^*$. Since φ is an automorphism of X, we have that $F \in \mathscr{E}$. We will now compute a basis for $\mathscr{E}$.

By Proposition 2.2, we have $\ell = n+1$. We now fix an order of the weights a so that we have $d = a_{n+1}m_{n+1} + a_0$ and $d = a_i m_i + a_{i+1}$, for all $i \in \{0, 1, \ldots, n\}$. Let now

$$\sigma_0 = 1, \quad \text{and} \quad \sigma_i = (-1)^i \prod_{t=1}^{i} \frac{d-a_t}{a_{t-1}}, \quad \text{for all } i \in \{1, 2, \ldots, n+1\}.$$

We denote the image of σ_i in $\mathbf{Z}/p\mathbf{Z}$ by the same letter σ_i. By Remark 2.3, with the chosen order of the weight, the automorphism φ of order prime p is given by the following signature

$$\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1}) \in (\mathbf{Z}/p\mathbf{Z})^{n+2}. \tag{10}$$

Let now $\mathbf{x}^r$ be a monomial in S, then

$$\mathbf{x}^r = x_0^{r_0} \cdots x_{n+1}^{r_{n+1}}, \quad \text{with} \quad \sum_{i=0}^{n+1} a_i r_i = d, \text{ and } r_i \in \mathbf{Z}_{\geq 0}.$$

Assume further that $\mathbf{x}^r \in \mathscr{E}$. Hence, by (10), we have

$$\sum_{i=0}^{n+1} \sigma_i r_i \equiv 0 \pmod{p} \tag{11}$$

By (9), we have

$$(-1)^{n+1} dp - 1 = \sigma_{n+1}\left(\frac{d-a_0}{a_{n+1}}\right)$$

Multiplying by a_{n+1} and taking the equation modulo p, we obtain

$$(d-a_0)\sigma_{n+1} \equiv -a_{n+1} \pmod{p} \tag{12}$$

Since $d - a_0$ is relatively prime to p, from (11) we have

$$(d-a_0)\sum_{i=0}^{n} \sigma_i r_i + (d-a_0)\sigma_{n+1} r_{n+1} \equiv 0 \pmod{p}$$

Replacing (12) in this equation and using that $\sigma_0 = 1$, we obtain

$$\left[(d-a_0)r_0 - a_{n+1}r_{n+1}\right] + (d-a_0)\sum_{i=1}^{n} \sigma_i r_i \equiv 0 \pmod{p}$$

Now, this last equality implies that $a_{n+1}r_{n+1} = k(d-a_0)$ and a straightforward computation shows that $k < d$. This yields

$$\big[r_0 - k\big] + \sum_{i=1}^{n} \sigma_i r_i \equiv 0 \pmod{p} \tag{13}$$

Remark that σ_i divides σ_{i+1} for every $i \in \{0,1,\ldots,n\}$. Hence, taking this equation modulo $\sigma_1 p$ we obtain:

$$r_0 - k \equiv 0 \pmod{\sigma_1 p}$$

Since r_0 and k are both bounded by $d < p$, we conclude $r_0 - k = 0$ yielding

$$(d - a_0) r_0 = a_{n+1} r_{n+1} \tag{14}$$

Now, taking (13) modulo $\sigma_2 p$ we obtain

$$\sigma_1 r_1 \equiv 0 \pmod{\sigma_2 p} \quad \text{wich yields} \quad r_1 = 0\,.$$

Recursively, taking (13) modulo $\sigma_i p$, for every $i \in \{3,4,\ldots,n-1\}$ we conclude that $r_i = 0$ for every $i \in \{1,2,\ldots,n\}$. And now (14) implies

$$r_0 = 1,\ \text{and}\ r_{n+1} = \frac{d-a_0}{a_{n+1}} = m_{n+1}, \quad \text{which yields} \quad \mathbf{x}^r = x_{n+1}^{m_{n+1}} x_0\,.$$

Remark that in (13), we used that σ_0 was chosen to be 1. To conclude, remark that taking a cyclic permutation of the weights a allows us to set any $\sigma_i = 1$ for the signature of the automorphism φ. Hence, we conclude that a basis for the eigenspace $\mathscr{E}$ is

$$\{x_{n+1}^{m_{n+1}} x_0\} \cup \{x_i^{m_i} x_{i+1} \mid i = 0,1,\ldots,n\}\,.$$

Hence,

$$F = \lambda_0 \cdot x_0^{m_0} x_1 + \lambda_1 \cdot x_1^{m_1} x_2 + \cdots + \lambda_n \cdot x_n^{m_n} x_{n+1} + \lambda_{n+1} \cdot x_{n+1}^{m_{n+1}} x_0.$$

Since $X = V(F)$ is quasi-smooth, by Lemma 1.3 , $\lambda_i \neq 0$, for all $i \in \{0,1,\ldots,n+1\}$ and applying a linear change of coordinates we can put

$$F = x_0^{m_0} x_1 + x_1^{m_1} x_2 + \cdots + x_n^{m_n} x_{n+1} + x_{n+1}^{m_{n+1}} x_0\,,$$

which proves that $X = V(F)$ is isomorphic to the Klein hypersurface. □

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